LOCALISATION OF THE DONALDSON'S INVARIANTS ALONG SEIBERG-WITTEN CLASSES.

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Abstract. This article is a first step in establishing a link between the Donaldson polynomials and Seiberg-Witten invariants of a smooth 4-manifold.

§0. Introduction.

As the result of analysis of the structure of the Donaldson's invariant of a smooth simply connected 4 manifold X Kronheimer and Mrowka [K-M1] (see also [F-S]) proved the existence of k_X classes

$$K_1, \dots, K_{k_X} \in H_2(X, \mathbb{Z})$$

and k_X polynomials

$$f_1, \dots, f_{k_X} \in \mathbb{Q}[z]$$

such that the Donaldson's invariant of X is defined uniquely by these pairs.

These classes are called the basic classes of X and the set $\{K_i\}$, for $i = 1, \dots, k_X$, in $H^2(X, \mathbb{Z})$ is a diffeomorphism invariant. So the diffeomorphism group $Diff X$ admits a representation to the symmetric group on k_X letters. A little bit later Seiberg and Witten introduced another set of classes $\{SW\}$ and polynomials and Witten predicted the shape of the Donaldson's invariant in terms of these data. S-W classes are labelling the moduli spaces $\{\mathcal{M}_{SW}\}$ of the solutions of the Seiberg-Witten system of equations (see §1) and the dimensions of these moduli spaces are the degrees of the polynomials. Using other type of system of equations we will prove that the classes of Kronheimer and Mrowka and Seiberg-Witten coincide. This paper is the first step of this proof and contains those facts from gauge theory which we need for the proof.

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§1. Configuration spaces and its cohomology.

Let $c \in H^2(X)$ be a $Spin^c$ structure on X, i.e. a pair of complex hermitian rank 2 bundles $W_c^{\pm}, c_1(W_c^{\pm}) = c$ (we shall omit the subscript c if there is no confusion). Then the tangent bundle

$$TX = Hom_h(W^-, W^+) \tag{1.1}$$

is the bundle of homomorphisms preserving the Hermitian structures.

Fix a pair of connections $\nabla^{\pm}$ on $W^{\pm}$ such that their tensor product is the Levi Civita connection on TX (which is the same as fixing the determinant $\nabla_{det} = det\nabla^{\pm}$). This defines a Dirac operator:

$$\mathcal{D}^{g,\nabla_{det}} : \Gamma(W^+) \to \Gamma(W^-) \tag{1.2}$$

as a composition of the connection $\nabla^+ : \Gamma(W^+) \to \Gamma(W^+ \otimes T^*X)$ and the convolution along (1.1). We shall omit the superscripts if this will not cause confusion.

U(1) - monopoles.

Let us recall that the configuration space $\mathcal{C}_{detW^+}$ for $U(1)$-Seiberg-Witten moduli space is

$$\mathcal{A}_{detW^+} \times \Gamma(W^+)/\mathcal{G}_{detW^+} = \mathcal{N} \times \Gamma(W^+)/S^1 \tag{1.3}$$

where $\mathcal{N} = ker(d^* : \Omega^1 \to \Omega^0)$ is a slice of an action of $\mathcal{G}_{detW^+}$. The Seiberg-Witten moduli spaces $\mathcal{M}_{SW}(0, c)$ are defined as the space of solutions of the system

$$\mathcal{D}^{\nabla_{det}}\phi = 0$$

$$F^+_{\nabla_{det}} = -(\phi \otimes \overline{\phi})_0, \tag{1.4}$$

modulo action of the gauge group, where $(\nabla_{det}, \phi) \in \mathcal{A}_{detW^+} \times \Gamma(W^+)$. Over the subspace $\mathcal{C}^*_{detW^+} = \mathcal{N} \times \mathbb{P}(\Gamma(W^+))$ of non-reducible points of $\mathcal{C}_E$ there is a complex linear bundle $\mathcal{O}_{\mathcal{C}^*}(1) = \mathcal{O}_{\mathbb{P}(\Gamma(W^+))}(1)$ with the first Chern class $t = c_1(\mathcal{O}_{\mathcal{C}^*}(1))$ generating the cohomology ring $H^*(\mathcal{C}^*, \mathbb{Q})$ in the case X is simply connected.

The system of equations (1.4) can be gauge invariantly perturbed as

$$\mathcal{D}^{\nabla_{det}}\phi = 0$$

$$(\omega + F_{\nabla_{det}})^+ = -(\phi \otimes \overline{\phi})_0, \tag{1.4_ω}$$

where ω is any 2-form.

The Seiberg-Witten moduli spaces $\mathcal{M}_{SW}(\omega, c)$ are defined as the space of solutions of the perturbed system $(1{,}4_\omega)$ modulo the action of the gauge group. For a generic form ω the moduli space is cut out transversely and therefore smooth.

We shall also consider another perturbation - perturbation of the metric g. The similar "generic position" statement for this space of parameters is provided by the following