POSITIVE CHARACTERISTIC CLASSES OF VECTOR SHEAVES ON ARITHMETIC VARIETIES

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1995

Abstract. Sufficient conditions for the existence of regular global sections of vector sheaves on arithmetic varieties are given. As an application, the intersection theoretic positivity of monomials in Chern classes of certain vector sheaves on arithmetic varieties is proved.

The main part of this work was done at the Max-Planck-Institut für Mathematik Bonn during the special activity on Arakelov theory 1994, the author's participation in which was made possible by U Jannsen, whom he wants to thank for this. Further thanks are due to T Zink and Bielefeld University, where this paper was completed. Moreover, the author is very grateful to L Moret-Bailly for his kind attention. Worth mentioning are also stimulating conversations with W Gubler.

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

Notation

The sign $\triangleright$ separates hypotheses and assertions. The symbols $\overset{\circ}{\subset}$ and $\overset{\circ}{\supset}$ denote **open** inclusions, $\operatorname{pt} X$ is the set of closed points of a topological space X, $\operatorname{cl}_B A$ is the integral closure of a ring A in a ring $B \supset A$.

1. Regular global sections of vector sheaves

1.0 Skolem's Theorem. *Let K be a number field, T a finite set of places of K, R the ring of T-integers of K, $S := \operatorname{Spec} R$. Let $n \in \mathbb{N}$ and $X \overset{\circ}{\subset} \mathbb{A}^n_R$ R-dense*
 $\triangleright$ *There exists a finite field extension $L \supset K$ such that $X(\operatorname{cl}_L R) \neq \emptyset$*

Proof. [M, Th. 1.6] □

Notation. Let $f\colon Y \to X$ be a morphism of schemes, and let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Then $\mathcal{F}|_Y$ means $f^*\mathcal{F}$ and $\Gamma(Y, \mathcal{F})$ means $\Gamma(\mathcal{F}|_Y)$.

1.1 Definition. Let X be a locally Noetherian scheme, $\mathcal{E}$ a vector sheaf on X, $s \in \Gamma(\mathcal{E})$, and $V(s)$ the zero subscheme of X defined by s.

s is *regular* iff for any irreducible component X' of X the condition

$$\operatorname{codim}_{X'}(V(s) \cap X') \geq \min\{\operatorname{rg}(\mathcal{E}|_{X'}), \dim X' + 1\}$$

is satisfied.

It is called *strictly regular* iff it is regular and for every irreducible component X' of X with $\dim X' \geq \operatorname{rg}(\mathcal{E}|_{X'})$ we have $V(s) \cap X' \neq \emptyset$ (hence $\operatorname{codim}_{X'}(V(s) \cap X') = \operatorname{rg}(\mathcal{E}|_{X'})$).

If $X \to Y$ is a morphism of schemes, $s \in \Gamma(\mathcal{E})$ is called *fiberwise (strictly) regular* iff for all $y \in Y$ the restricted section $s_y := s|_{X_y} \in \Gamma(X_y, \mathcal{E})$ is (strictly) regular.

1.2 Remark. If X is not Cohen–Macaulay, this definition of regularity of sections differs from Fulton's [F, B 3.4].

1.3 Lemma. *Let S be as above, X an irreducible scheme, and $f\colon X \to S$ a flat morphism of finite type and pure relative dimension d. Let $\mathcal{E}$ be a vector sheaf on X, $s \in \Gamma(\mathcal{E})$.*

(1) *s is fiberwise regular* $\triangleright$ *s is regular*

(2) *Suppose that $r := \operatorname{rg}\mathcal{E} \neq d+1$, s is regular, and if $r \leq d$ there exists an $y \in S$ with s_y strictly regular.* $\triangleright$ *s is strictly regular*

Proof. ad (1): If $V(s) = \emptyset$ there is nothing to prove. Otherwise let $Z \neq \emptyset$ be an irreducible component of $V(s)$, X' the irreducible component of X with $X' \supset Z$, $n := \operatorname{rg}(\mathcal{E}|_{X'})$. Then $\operatorname{codim} Z \leq n$. Let $x \in \operatorname{pt} Z$, $y := f(x)$. $Z_y \neq \emptyset$ implies $n \leq \dim X_y$, and

$$\dim_x Z_y = \dim_x X_y - \operatorname{rg}_x \mathcal{E}|_{X_y} = \dim X_y - n$$

In this situation [EGA IV.14.5.1] teaches us that $\operatorname{codim} Z = n$, hence s is regular.