ON THE DERIVATIVE OF THE STABLE HOMOTOPY OF MAPPING SPACES

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Introduction

In this paper we shall provide a homotopy-theoretic computation of the *derivative*, in the sense of Goodwillie [G], of the functor

$$X \quad \longmapsto \quad Q_+ F(K, X) \ ,$$

where K denotes a simplicial complex, $F(K, X)$ denotes the space of unpointed maps $K \to X$, and Q_+ denotes unreduced stable homotopy.

The derivative of this functor was first determined using framed bordism theory in [loc.cit.]; the answer was expressed as the section space of a certain fibration over K. The first homotopy theoretic computation of it appeared in [H] and independently in [A], using configuration spaces. However, these proofs also used manifolds at various places in the argument.

We shall make the computation by a different route which avoids the use of configuration spaces and manifolds. The main idea is to apply a special case of the *chain rule* of [KR] together with certain elementary results about homotopy limits. One advantage of our approach is that it generalizes to give a determination of the derivative of

$$X \quad \longmapsto \quad \Omega^\infty \mathbf{E} \wedge_{hH} F(K, X) \ ,$$

where H is a topological group, K is a space with pointed H-action, and $\mathbf{E}$ is a spectrum with H-action (see 6.2 below).

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1. Preliminaries

1.1. In this paper **Top** denotes category of topological spaces, $\mathbf{Top}_*$ is the category of pointed spaces, and **Sp** denotes the category of (omega) spectra. We will implicity assume that all spaces appearing below are the homotopy type of a CW complex.

If Z is a pointed space, and $\mathbf{E}$ is a spectrum, then $F(Z, \mathbf{E})$ denotes the function spectrum whose n-th space is the space of based maps from Z to the n-th space of $\mathbf{E}$. If $\mathbf{E}$ is a spectrum equipped with action by a topological group G, we let $\mathbf{E}_{hG}$ denote the associated homotopy orbit spectrum defined by $(\mathbf{E} \wedge EG_+)/G$.

We cite some results about homotopy limits in closed model categories. Let $\mathbf{C}$ be a closed model category. We will assume that $\mathbf{C}$ admits a functorial cylinders. One therefore has the notion of *homotopy equivalence* in $\mathbf{C}$.

If $f\colon D \to \mathbf{C}$ denotes a finite diagram (i.e. a covariant or contravariant functor from a very small category D into $\mathbf{C}$), it makes sense to speak of the *homotopy limit* and *homotopy colimit of* f (see e.g., [DS,10.13]).

We denote

$$H^\bullet(D; f) \quad := \quad \operatorname*{holim}_D f \ .$$

and

$$H_\bullet(D; f) \quad := \quad \operatorname*{hocolim}_D f \ ,$$

to emphasize the analogy with (co)-homology with coefficients in a sheaf.

We shall say that a natural transformation of $f \to g$ functors is a *natural weak equivalence* if $f(d) \to g(d)$ is a (weak) equivalence of $\mathbf{C}$. More generally, we say that f and g are *naturally weak equivalent*, if there exists a finite chain of natural weak equivalences,

$$f = f_0 \xleftarrow{\simeq} f_1 \xrightarrow{\simeq} \ldots \xleftarrow{\simeq} f_n = g \ .$$

We have,

Lemma 1.2. *(1). Suppose that $f(d)$ and $g(d)$ are fibrant for all $d \in D$. If $f \to g$ is a natural weak equivalence of functors $D \to \mathbf{C}$ then the induced map*

$$H^\bullet(D; f) \to H^\bullet(D; g)$$

is a homotopy equivalence of $\mathbf{C}$.

(2). If f and g are naturally weak equivalent, then $H^\bullet(D; f)$ and $H^\bullet(D; g)$ are connected by a chain of homotopy equivalences of $\mathbf{C}$. In particular, they are weak equivalent.

Proof. This is shown in [BK,XI§6] when $\mathbf{C}$ is the category of simplicial sets. The proof in the present context is given by direct generalization. $\square$

In this paper, we will take $\mathbf{C}$ to be the category of spectra. The category of spectra can be equipped with a closed model structure (see e.g., [BF]). The cautious reader may alternatively work in $\mathbf{Top}_*$ (with closed model structure determined by Serre fibrations and weak homotopy equivalences), if she/he so wishes, to obtain the theorem on the level of zero-th spaces.