Finite dimensional approximation of diffusion processes on infinite dimensional spaces

M. Röckner *   T. S. Zhang *

Abstract

We prove that the laws of diffusion processes $\mathbf{M}$ on E associated with Dirichlet forms of type $\mathcal{E}(u,v) = \int_E \langle A(z)\nabla u(z), \nabla v(z)\rangle_H \mu(dz)$, where H, E are separable Hilbert spaces, are the weak limits of laws of finite dimensional diffusions. These are associated with the image Dirichlet forms obtained from $\mathcal{E}$ under projections from E onto finite dimensional subspaces in H. As a by-product we obtain Hoelder continuity of the sample paths as well as a new existence proof for the infinite dimensional diffusion $\mathbf{M}$.

AMS Subject Classification
Primary: 60J60
Secondary: 31C25
Key words: Infinite dimensional diffusion processes; Dirichlet forms; martingale decomposition, tightness
Running head: Approximation of diffusions

*Fakultät für Mathematik, Universität Bielefeld, Postfach 100131, 33501 Bielefeld, Germany.

1 Introduction, framework and main results

The purpose of this paper is to construct natural finite dimensional approximations to infinite dimensional stochastic processes associated with Dirichlet forms of diffusion type (cf. e.g. [AR91]). Though the existence of such infinite dimensional diffusions is well-known (see [Sch90], [RS92] or [MR92, Chap.IV, Subsection 4b]), our approximation procedure also gives a new existence proof (at least if the state space is a Hilbert space and condition (1.1) below holds). Apart from general interest these approximations are motivated since they can be used to obtain *new* information about the limiting process. For example, we can now prove Hoelder continuity of its sample paths. To state and illustrate our results and their applications more precisely, as well as to give an overview of the organization of this paper we need some preparations. We start with describing our framework.

Let H, E be separable Hilbert spaces (with inner product $\langle\ ,\ \rangle_H$, $\langle\ ,\ \rangle_E$ respectively), $H \subset E$ densely and continuously, and such that, in addition,

$$\text{the embedding } H \subset E \text{ is Hilbert-Schmidt.} \tag{1.1}$$

Let μ be a probability measure on the Borel σ-algebra $\mathcal{B}(E)$ of E such that $E' \subset L^2(E;\mu)$. Here E' denotes the (topological) dual of E. Define the linear space of smooth bounded cylinder functions, as usual, by

$$\mathcal{F}C_b^\infty := \{f(l_1,\dots,l_m) \mid m \in \mathbb{N},\ f \in C_b^\infty(\mathbb{R}^m),\ l_1,\dots l_m \in E'\}\ .$$

Here $C_b^\infty(\mathbb{R}^m)$ is the space of all bounded infinitely differentiable real-valued functions on $\mathbb{R}^m$ with all partial derivatives bounded.

Let $\mathcal{L}^\infty(H)$ denote the space of all bounded linear operators on H with usual operator norm $\|\ \ \|_{\mathcal{L}^\infty(H)}$. Let $A : E \to \mathcal{L}^\infty(H)$ be a strongly measurable map such that $A(z)$ is self-adjoint and positive definite for all $z \in E$ and

$$\text{there exists } q \in]1,\infty[\text{ such that } \int \|A(z)\|^q_{\mathcal{L}(H)}\ \mu(dz) < \infty\ . \tag{1.2}$$

Consider the following symmetric positive definite bilinear form on $L^2(E;\mu)$ $(:= L^2(E \to \mathbb{R};\mu))$

$$\mathcal{E}(u,v) := \int_E \langle A(z)\nabla u(z), \nabla v(z)\rangle_H\ \mu(dz) \quad ;\ u,v \in \mathcal{F}C_b^\infty\ . \tag{1.3}$$