DIRICHLET FORMS AND MARKOV PROCESSES: A GENERALIZED FRAMEWORK INCLUDING BOTH ELLIPTIC AND PARABOLIC CASES

WILHELM STANNAT

Universität Bielefeld, Fakultät für Mathematik,
Postfach 100131, 33501 Bielefeld, Germany

ABSTRACT. We extend the framework of classical Dirichlet forms to a class of bilinear forms, called generalized Dirichlet forms, which are the sum of a coercive part and a linear unbounded operator as a perturbation. The class of generalized Dirichlet forms, in particular, includes symmetric and coercive Dirichlet forms (cf. [Fu2], [M/R]) as well as time dependent Dirichlet forms (cf. [O1]) as special cases and also many new examples. Among these are, e.g. transformations of time dependent Dirichlet forms by α–excessive functions h (h–transformations), Dirichlet forms with time dependent linear drift and fractional diffusion operators. One of the main results is that we identify an analytic property of these forms which ensures the existence of associated strong Markov processes with nice sample path properties, and give an explicit construction for such processes. This construction extends previous constructions of the processes in the elliptic and the parabolic cases, is, in particular, carried out on general topological state spaces (as in [M/R]), and is applied to the above examples.

Mathematics Subject Classification (1991): 60J, 31C25

Running Head: Generalized Dirichlet forms

Key Words: Generalized Dirichlet forms, Strong Markov processes, Quasi–regularity

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Diese Arbeit ist eingereicht bei: Potential Analysis

I. Introduction

This paper is concerned with the development of an extended framework for Dirichlet forms including both elliptic cases (i.e.,symmetric and coercive Dirichlet forms as treated in [Fu2] resp. [M/R]) and parabolic ones (i.e.,time dependent Dirichlet forms as considered in [O1]) .

To this end we introduce a new class of bilinear forms, defined on some L^2-space $L^2(E,m)$, called *generalized Dirichlet forms*, which are in general neither symmetric nor coercive nor of the type considered in [O1], but still have corresponding generators, strongly continuous contraction semigroups and resolvents. These forms are the sum of a coercive closed form $(\mathcal{A},\mathcal{D}(\mathcal{A}))$ (in the sense of [M/R]) and a linear (unbounded) operator (Λ, D) on $L^2(E,m)$, where the latter is a small perturbation of the first in the following sense: (Λ, D) generates a C_0–semigroup $(U_t)_{t\geq 0}$ of contractions such that $(U_t)_{t\geq 0}$ and the adjoint semigroup $(\hat{U}_t)_{t\geq 0}$ can be extended to C_0–semigroups on the dual space $D(\mathcal{A})'$ of $D(\mathcal{A})$. Then the generalized bilinear form is defined as follows:

$$\mathcal{E}(u,v) := \begin{cases} \mathcal{A}(u,v) - \langle \Lambda u, v\rangle & \text{if } u \in D(\Lambda)\cap D(\mathcal{A}), v \in D(\mathcal{A}) \\ \mathcal{A}(u,v) - \langle \hat{\Lambda} v, u\rangle & \text{if } u \in D(\mathcal{A}), v \in D(\hat{\Lambda})\cap D(\mathcal{A}) \ . \end{cases} \tag{1.1}$$

Here $\langle\cdot,\cdot\rangle$ denotes the dualization between $D(\mathcal{A})$ and $D(\mathcal{A})'$, and $(\Lambda, D(\Lambda))$, $(\hat{\Lambda}, D(\hat{\Lambda}))$ the generators of the extension of $(U_t)_{t\geq 0}$, $(\hat{U}_t)_{t\geq 0}$ resp. Forms of this type have been studied in [L/Ma] and also in [L].

We are interested in a special subclass of these forms, called *generalized Dirichlet forms*, and their probabilistic counterparts. We call a form of type (1.1) a generalized (semi–) Dirichlet form if the associated strongly continuous contraction resolvent $(G_\alpha)_{\alpha>0}$ is *sub–Markovian* and if there is a dense subspace $\mathcal{Y}\subset D(\Lambda)\cap D(\mathcal{A})$ such that $u\wedge\alpha\in\mathcal{Y}$ for $u\in\mathcal{Y}$ and $\alpha\geq 0$. It is clear that in the case $\Lambda = 0$ we end up in the class of coercive Dirichlet forms (cf. [M/R],[M/Ov/R]). But the class of generalized Dirichlet forms also includes the time dependent Dirichlet forms, as considered in [O1], which are defined as follows: Let $(\mathcal{E}^{(t)}, V)_{t\in\mathbb{R}}$ be a family of coercive Dirichlet forms with uniform sector constant K and common domain V in some L^2–space such that $t\mapsto\mathcal{E}^{(t)}(u,v)$ is Borel–measurable, $\forall u,v\in V$, and $c\mathcal{E}^{(0)}_1(u,u)\leq\mathcal{E}^{(t)}_1(u,u)\leq c^{-1}\mathcal{E}^{(0)}_1(u,u)$, $\forall u\in V$, $t\in\mathbb{R}$, for some positive constant c. Then the corresponding *time dependent Dirichlet form*

$$\mathcal{E}(u,v) := \begin{cases} \int_{\mathbb{R}}\mathcal{E}^{(t)}(u(t),v(t))\,dt - \int_{\mathbb{R}}\langle\frac{du}{dt}(t), v(t)\rangle\,dt & \text{if } u\in L^2(\mathbb{R},V)\cap \\ & H^{1,2}(\mathbb{R},V'), v\in L^2(\mathbb{R},V) \\ \int_{\mathbb{R}}\mathcal{E}^{(t)}(u(t),v(t))\,dt + \int_{\mathbb{R}}\langle\frac{dv}{dt}(t), u(t)\rangle\,dt & \text{if } u\in L^2(\mathbb{R},V), v\in \\ & L^2(\mathbb{R},V)\cap H^{1,2}(\mathbb{R},V'), \end{cases}$$

is a generalized Dirichlet form in the above sense. Here $\langle\cdot,\cdot\rangle$ denotes the dualization between V and the dual space V'. We emphasize, however, that if we replace the time–derivative in these examples by the time derivative $(\frac{d}{dt})^\alpha$ of power α, $\alpha\in(0,1)$, we still have a generalized Dirichlet form (cf. Example II.11(iv)). Moreover,