Quantum Serre Relations

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Serre has exhibited a presentation of the finite dimensional semisimple complex Lie algebras by generators and relations in 1966. This presentation uses only the data given by the Cartan matrix of the Lie algebra. Of great importance was the introduction of correspondingly defined Lie algebras by Kac and Moody in 1968, starting with a generalized Cartan matrix or, equivalently, with a suitable bilinear form on a free abelian group $\mathbb{Z}I$, where I is some finite set. Presentations of a Lie algebra $\mathbf{g}$ by generators and relations are also presentations of the universal enveloping algebra $U(\mathbf{g})$; of course, one has to rewrite the Lie bracket operation in terms of commutators.

The quantum groups as investigated by Drinfeld and Jimbo in 1985 are associative algebras which are presented by relations similar to the Serre relations used in the definition of a Kac-Moody algebra. Here, an additional parameter, say v, is involved: the binomial coefficients are replaced by the corresponding Gauß polynomials in the variable v. Even the most enthusiastic mathematician will not have foreseen the big variety of connections between these algebras and quite different areas of mathematics and physics which have been encountered in the meantime.

Now, let us consider in more detail the positive part U^+ of such a quantum group U. Using the representation theory of finite dimensional hereditary algebras, one can realize U^+ in different ways, as a sort of Grothendieck ring of perverse sheaves [L] or as a twisted generic composition algebra [G]. Closely related algebras such as Hall algebras [R1,G] should be treated at the same time. Lusztig [L] has stressed the importance of algebraic structures A which are similar to bialgebras, but where multiplication and comultiplication are compatible only up to a twist on $A \otimes A$. Enlarging his considerations, we may start with a pair χ of bilinear forms on the free abelian group $\mathbb{Z}I$ and consider what we call (K, v, χ)-bialgebras. Algebras such as U^+, the Hall algebras, and also the free K-algebra $K\langle I\rangle$ generated by I may be considered as (K, v, χ)-bialgebras. Recall that U^+ can be defined as the factor algebra of $K\langle I\rangle$ modulo the radical of the Lusztig form, this is a bilinear form which is uniquely determined by the (K, v, χ)-bialgebra structure of $K\langle I\rangle$.

The present note will consider the Lusztig form in the general case when χ is an arbitrary pair of bilinear forms. Elements of $K\langle I\rangle$ which are similar to quantum Serre relations will be shown to belong to the left radical of the Lusztig form, provided certain integrality conditions are satisfied. So we recover the result that U^+, defined as the factor algebra of $K\langle I\rangle$ modulo the radical of the Lusztig form, satisfies the quantum Serre relations ([L], Proposition 1.4.3). The general form of the quantum Serre relations as exhibited below occurs quite naturally as the fundamental relations of the Hall algebras [R2]. Similar considerations produce elements in the right radical of the Lusztig form.

1. (K, v, χ)-bialgebras.

The data which are given are as follows: Let K be a commutative ring and $v \in K$ an invertible element. Let I be a set. We denote by $\mathbb{Z}I$ the free abelian group with basis I, and χ', χ'' are two bilinear forms on $\mathbb{Z}I$ with values in $\mathbb{Z}$.

Let $A = \bigoplus_{x \in \mathbb{Z}I} A_x$ be a $\mathbb{Z}I$-graded K-algebra. Given a bilinear form ϕ on $\mathbb{Z}I$ and an invertible element $v \in K$, we denote by $A_{[v,\phi]}$ the algebra obtained from A by twisting the muliplication using v^{ϕ}; to be precise: the new multiplication $*$ is defined on the same underlying K-module $\bigoplus_{x \in \mathbb{Z}I} A_x$ by

$$a * b = v^{\phi(x,y)}\, ab,$$

where a, b are homogeneous elements of degree $x, y \in \mathbb{Z}I$ respectively. Note that $A_{[v,\phi]}$ is again a K-algebra [R3], and of course also $\mathbb{Z}I$-graded.

Let A be a $\mathbb{Z}I$-graded algebra. Given the pair $\chi = (\chi', \chi'')$ of bilinear forms on $\mathbb{Z}I$ and $v \in K$, we consider a corresponding map $(\mathbb{Z}I)^4 \to \mathbb{Z}$, which we also denote by χ and which is defined as follows:

$$\chi(x_1, x_2, x_3, x_4) = \chi'(x_1, x_4) + \chi''(x_2, x_3).$$

Note that this map χ is a bilinear form on $(\mathbb{Z}I)^2$ (the maps $(\mathbb{Z}I)^4 \to \mathbb{Z}$ obtained in this way may be characterized by certain bilinearity properties, see [R4]). We may consider $A \otimes A$ as a $(\mathbb{Z}I)^2$-graded algebra, where for $x, y \in \mathbb{Z}I$, we have $(A \otimes A)_{(x,y)} = A_x \otimes A_y$. Thus, given a pair $\chi = (\chi', \chi'')$ of bilinear forms on $\mathbb{Z}I$, we may use the bilinear form χ on $(\mathbb{Z}I)^2$ in order to twist the multiplication of $A \otimes A$ and we obtain in this way the algebra $(A \otimes A)_{[v,\chi]}$ with multiplication

$$(a_1 \otimes a_2) * (b_1 \otimes b_2) = v^{\chi'(a_1,b_2)+\chi''(a_2,b_1)}\, a_1 b_1 \otimes a_2 b_2.$$

Here we see in which way the two bilinear forms are used: the form χ'' draws attention to the interchange of the inner elements a_2, b_1, whereas the form χ' is called in for the outer elements a_1 and b_2.

By definition, a (K, v, χ)-*bialgebra* is of the form $A = (A, \mu, \delta)$, where A is a K-module with a direct decomposition $A = \bigoplus_{x \in \mathbb{Z}I} A_x$, such that (A, μ) is a $\mathbb{Z}I$-graded algebra, (A, δ) is a $\mathbb{Z}I$-graded coalgebra and such that on the one hand, the counit ϵ satisfies $\epsilon(1) = 1$, and, on the other hand,

$$\delta \colon A \to (A \otimes A)_{[v,\chi]}$$

is an algebra homomorphism.

2. The free K-algebra $K\langle I\rangle$ as (K, v, χ)-bialgebra.

Let $F = K\langle I\rangle$ be the free K-algebra generated by I; we may consider it as the semigroup algebra of the free semigroup $\langle I\rangle$ generated by I. The generator corresponding to $i \in I$ will be denoted by θ_i. Thus, the elements of $\langle I\rangle$ are words in