Normal subgroups of $\mathbf{SL}_{1,D}$ and the classification of finite simple groups

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1. Introduction

Let G be a simple simply connected algebraic group over an algebraic number field K. There is a conjecture due to Platonov and Margulis (*cf.* [PR3], ch. IX) that the normal subgroup structure of the group of K-rational points $G(K)$ should be as follows:

(PM) *let T be the set of all non-archimedean places v of K such that G is K_v-anisotropic; then for any noncentral normal subgroup $N \subset G(K)$ there should be an open normal subgroup $W \subset G(K,T) = \prod_{v\in T} G(K_v)$ such that $N = G(K) \cap W$; in particular, if $T = \emptyset$ (which is always the case if G is not of type A_n) then $G(K)$ should not have any proper noncentral normal subgroups (i.e. should be projectively simple).*

By now, this conjecture has been proved for almost all K-isotropic groups and most K-anisotropic groups of type different from A_n (*cf. loc. cit*). So, major attention is currently focused on the main remaining case of anisotropic groups of type A_n. The first result dealing with (PM) for this case was obtained in [PR1]; it was shown that for $G = \mathbf{SL}_{1,D}$, D a quaternion division algebra such that $T = \emptyset$, the group $G(K)$ is perfect (i.e. $G(K) = [G(K), G(K)]$). Note that for $G = \mathbf{SL}_{1,D}$, the set T introduced in the statement of (PM) coincides with the set of non-archimedean places v of K such that $D_v = D \otimes_K K_v$ is a division algebra. Using some methods from [PR1], Margulis [M2] proved (PM) for $G = \mathbf{SL}_{1,D}$ where D is an arbitrary quaternion algebra, in full. The technique of [PR1] was extended in [PR2] to division algebras of arbitrary index to prove (under some minor restriction on the dyadic ramification places of D) that the commutator subgroup $[G(K), G(K)]$ is always T-adically open in $G(K)$ (by the T-adic topology we mean the topology induced from the group $G(K,T) = \prod_{v\in T} G(K_v)$). Subsequently, Raghunathan [Ra] lifted the restriction imposed on D in [PR2] and proved that, more generally, the commutator subgroup $[U, U]$ of any T-adically open subgroup $U \subset G(K)$ is again T-adically open. Finally, Tomanov [To] gave reduction of (PM) for $G = \mathbf{SL}_{1,D}$ to division algebras of odd index; in particular he proved (PM) for all algebras of index 2^d. In the present paper we prove (PM) for $G = \mathbf{SL}_{1,D}$ where D is a division algebra of index 3. This is the first result on (PM) for division algebras of odd index, and by Tomanov's Theorem it implies (PM) also for all division algebras of index $3 \cdot 2^d$.

Theorem 1.1 *Let D be a central division algebra over K of index 3, $G = \mathbf{SL}_{1,D}$. Then the normal subgroup structure of $G(K)$ is given by* (PM).

The proof of this theorem is based on methods which are essentially different from those employed in the previous works on (PM). Its main feature is the usage of (CSG), the classification of finite simple groups. (By (CSG) we mean the statement that any finite non-abelian simple group is either a group of Lie type or an alternating group, or else is one of the 26 sporadic groups, *cf.* [Atlas]). To be more precise, (CSG) is used at two places in the proof of Theorem 1.1. One of them is so-called Push-up Theorem (*cf.* Theorem 2.1) which claims that if for $G = \mathbf{SL}_{1,D}$, D a division algebra of arbitrary index, the group $G(K)$ fails to satisfy (PM) then there exists a normal subgroup $N \subset D^*$ such that the quotient D^*/N is a finite simple group. Actually, the proof of this assertion uses not (CSG) itself but its following important consequence (Theorem 2.4): any class-preserving automorphism of a finite simple group F is inner.

This result suggests the following conjecture which can be regarded as a generalization of (PM) to division algebras over arbitrary field: *let D be a finite dimensional algebra over an arbitrary field; then D^* does not have any normal subgroup N such that D^*/N is a non-abelian finite simple group.*

In the present paper, we prove this conjecture for algebras of degree 2 and 3, which yields Theorem m 1.1 and gives an alternate proof of Margulis' Theorem [M2]. The argument for algebras of degree 2 actually imitates the proof of Theorem 3 in [PR1] and does not use (CSG), while consideration of algebras of degree 3 is based on a new approach.

First, we show that all quotients of D^*, D a division algebra of degree 3, satisfy a certain abstract property (called here property (C), *cf.* §3), and then prove that, on the contrary, none of the (known) finite simple groups has this property (*cf.* Theorem 3.5). The proof of Theorem 3.5 takes the major part of this paper, and it is the other spot in the proof of Theorem 1.1 which depends on (CSG) (note that here the dependence on (CSG) is stronger than in the proof of the Push-up Theorem, viz. to handle the sporadic groups, we use the detailed information about centralizers of elements from [Atlas]).

We hope to apply the technique of this paper also to other division algebras. Apparently, this can be done via a more thorough analysis of "additive" properties of multiplicative normal subgroups in D^* of finite index. (Some important results in this direction were obtained in [BS], [Tu].) So, in §7 we describe the additive properties that would enable us to prove (PM).

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