Bounds for Determinants of Perturbed M-Matrices

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Abstract

A sharp upper bound for the relative error of the determinant of an M-matrix in terms of the relative errors of the matrix entries is given. It improves and completes a bound derived by Jungong and Jiang in [2].

1 Introduction, formulation of the result

Let A be an $n \times n$ M-matrix,

$$A = D - N = (a_{ij}), \tag{1}$$

where $D = diag(a_{ii}), N \geq 0$ with zero diagonal. Then

$$\gamma = \rho(D^{-1}N) < 1. \tag{2}$$

Here ρ denotes the Perron root of the nonnegative matrix $D^{-1}N$. We consider perturbations δA of A with small relative errors $\epsilon > 0$, i.e.

$$|\delta A| \leq \epsilon |A| = \epsilon(D + N). \tag{3}$$

Here we have used the notation $|B|$ for the matrix with the entries $|b_{ij}|$ if $B = (b_{ij})$. In the paper [2] the authors derive bounds for the relative error of the inverse of A in terms of γ and ϵ. They describe a function $c(\gamma, \epsilon)$ satisfying

$$|A^{-1} - (A + \delta A)^{-1}| \leq (c(\gamma, \epsilon) + O(\epsilon^2))A^{-1}. \tag{4}$$

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In the course of the proof a bound for the relative error of the determinant is needed. It is shown there that

$$\frac{\det(A+\epsilon|A|)}{\det(A)} \leq 1 + \frac{n\epsilon}{1-\gamma} + O(\epsilon^2). \tag{5}$$

In this note we improve this result and give *sharp* bounds, also depending on γ and ϵ only.

For stating our result in a short way we introduce the function

$$b(\gamma, t) = 1 + 2t\frac{1+\gamma^2}{1-\gamma^2} + t^2 \tag{6}$$

and observe that this function is positive for $t > -\frac{1-\gamma}{1+\gamma}$.
The main result of this note is the following

Theorem 1 *Let $A = D - N$, where $D = diag(a_{ii})$, be an $n \times n$ M-matrix, $\gamma = \rho(D^{-1}N) < 1$, $\epsilon > 0$ and $|\delta A| \leq \epsilon|A|$. If $\epsilon < \frac{1-\gamma}{1+\gamma}$ then for n even*

$$b(\gamma, -\epsilon)^{n/2} \leq \frac{\det(A+\delta A)}{\det(A)} \leq b(\gamma, \epsilon)^{n/2} \tag{7}$$

while for n odd

$$(1-\epsilon)b(\gamma, -\epsilon)^{(n-1)/2} \leq \frac{\det(A+\delta A)}{\det(A)} \leq (1+\epsilon)b(\gamma, \epsilon)^{(n-1)/2} \tag{8}$$

The bounds are sharp.

2 Proof

The proof of Theorem 1 will be an easy consequence of the following two results which, though serving mainly as lemmata for Theorem 1, are interesting in themselves and hence are also addressed as Theorems.

Theorem 2 *Under the assumptions of Theorem 1 let $F = \epsilon|A|$. If $\epsilon < 1$ then for n even*

$$\frac{\det(A+F)}{\det(A)} \leq b(\gamma, \epsilon)^{n/2} \tag{9}$$

while for n odd

$$\frac{\det(A+F)}{\det(A)} \leq (1+\epsilon)b(\gamma, \epsilon)^{(n-1)/2} \tag{10}$$

holds.
This bound is sharp for all n.