Acyclic and Oriented Chromatic Numbers of Graphs

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Abstract

The oriented chromatic number $\chi_o(\vec{G})$ of an oriented graph $\vec{G} = (V, A)$ is the minimum number of vertices in an oriented graph $\vec{H}$ for which there exists a homomorphism of $\vec{G}$ to $\vec{H}$. The oriented chromatic number $\chi_o(G)$ of an undirected graph G is the maximum of the oriented chromatic numbers of all the orientations of G. This paper discusses the relation between the oriented chromatic number and the acyclic chromatic number and some other parameters of a graph. We shall give a lower bound for $\chi_o(G)$ in terms of $\chi_a(G)$.

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An upper bound for $\chi_o(G)$ in terms of $\chi_a(G)$ was given by Raspaud and Sopena. We also give an upper bound for $\chi_o(G)$ in terms of the maximum degree of G. This upper bound improves an earlier upper bound of the same kind. We shall show that this upper bound is not far from being optimal.

1 Introduction

A homomorphism f of a digraph $\vec{G} = (V, A)$ to a digraph $\vec{H} = (V', A')$ is a mapping of V to V' such that for any arc (x, y) of $\vec{G}$, the image $(f(x), f(y))$ is an arc of $\vec{H}$. Homomorphisms of undirected graphs are defined similarly. Homomorphisms of digraphs and undirected graphs have been studied as a generalization of graph coloring in the literature [6, 7, 8, 13]. It is easy to see that a k-coloring of an undirected graph G is equivalent to a homomorphism of G to the complete graph K_k. Therefore the chromatic number $\chi(G)$ of an undirected graph G is equal to the minimum integer k such that there is a homomorphism of G to K_k, or equivalently $\chi(G)$ is equal to the minimum number of vertices of an undirected graph H for which there exists a homomorphism of G to H.

We call those digraphs which contain no opposite arcs *oriented graphs*. Generalizing the above definition of chromatic number to oriented graphs, we define the oriented chromatic number of an oriented graph $\vec{G}$ to be the minimum number of vertices in an oriented graph $\vec{H}$ for which there exists a homomorphism of $\vec{G}$ to $\vec{H}$. We denote this number by $\chi_o(\vec{G})$.

The oriented chromatic number of an oriented graph $\vec{G}$ can also be defined as follows: For an integer k, a k-coloring of $\vec{G}$ is an assignment c of $\vec{G}$ such that for every arc (x, y) of $\vec{G}$ the following is true: (1) $c(x) \neq c(y)$; (2) there is no arc (z, t) of $\vec{G}$ for which $c(t) = c(x)$ and $c(z) = c(y)$. The oriented chromatic number $\chi_o(\vec{G})$ of $\vec{G}$ is the minimum k for which there exists a k-coloring of $\vec{G}$.

For an undirected graph G, let $\mathcal{O}_G$ be the set of all orientations of G. We define the oriented chromatic number $\chi_o(G)$ of G to be the maximum of $\chi_o(\vec{G})$ over all orientations $\vec{G}$ of G, i.e., $\chi_o(G) = \max\{\chi_o(\vec{G}) : \vec{G} \in \mathcal{O}_G\}$.