On the dimensions of anisotropic quadratic forms with trivial invariants

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Abstract. Let WF denote the Witt ring of a field F of characteristic $\neq 2$ and let I^nF denote the n-th power of the ideal IF of even-dimensional forms in WF. The Arason-Pfister Hauptsatz states that if $0 \neq \varphi \in I^nF$ is anisotropic then $\dim \varphi \geq 2^n$. Pfister also showed that if $\varphi \in I^3F$ is anisotropic and $\dim \varphi > 8$ then $\dim \varphi \geq 12$. We extend this result to I^4F and show that if $\varphi \in I^4F$ is anisotropic and $\dim \varphi > 16$ then $\dim \varphi \geq 24$ and we provide some results on anisotropic 24-dimensional forms in I^4F.

1. Introduction

Let WF denote the Witt ring of a field F of characteristic $\neq 2$ and let I^nF denote the n-th power of the ideal IF of even-dimensional forms in WF. An important problem in the algebraic theory of quadratic forms is the characterization of those forms which lie in I^nF. It is well-known and easy to show that I^nF is additively generated by n-fold Pfister forms $\langle\langle a_1, \cdots, a_n\rangle\rangle = \langle 1, a_1\rangle \otimes \cdots \langle 1, a_n\rangle$. We will denote the set of all forms over F which are isometric (resp. similar) to n-fold Pfister forms by P_nF (resp. GP_nF). One possible approach to characterize forms in I^nF is the use of invariants, for example the signed discriminant of a form φ which is denoted by $d_{\pm}\varphi$ and which is an element in $\dot{F}/\dot{F}^2$ (here, $\dot{F} = F \setminus \{0\}$), or the Clifford invariant $c(\varphi)$ which lies in ${}_2BrF$, the 2-torsion part of the Brauer group of F. One readily checks that $I^2F = \{\varphi \in IF \,|\, d_{\pm}\varphi = 1\}$. A consequence of Merkuriev's celebrated theorem in [M] is that $I^3F = \{\varphi \in I^2F \,|\, c(\varphi) = 1\}$.

In order to extend these results it turned out to be useful to apply cohomological methods which was first done systematically by Arason in [A]. Let H^nF denote the Galois cohomology group $H^n(\mathrm{Gal}(F_s/F), \mathbb{Z}/2)$ where F_s is a separable closure of F. Arason showed that there are well-defined maps $P_nF \longrightarrow H^nF$ which, for $n \leq 3$, extend to well-defined homomorphisms

$$\bar{e}^n_F \,:\, I^nF/I^{n+1}F \longrightarrow H^nF\ .$$

Jacob and Rost [JR] and Szyjewski [Sz] have proved that Arason's map extends to a well-defined homomorphism $\bar{e}^n_F$ as above also for $n = 4$. In fact, one easily sees that $\bar{e}^0_F$ is nothing but the map given by dimension modulo 2 using $\mathbb{Z}/2 \simeq H^0F$, and $\bar{e}^1_F$ is induced by the signed discriminant via $\dot{F}/\dot{F}^2 \simeq H^1F$. $\bar{e}^2_F$ is induced by the Clifford invariant map using the fact that ${}_2BrF \simeq H^2F$. An easy check shows that $\bar{e}_F^0$ and $\bar{e}_F^1$ are isomorphisms. It is Merkuriev's theorem which says that $\bar{e}_F^2$ is an isomorphism as well. Merkuriev and Suslin [MSu] and independently Rost [R1] showed that $\bar{e}_F^3$ is also an isomorphism. The conjecture is that Arason's map extends to an isomorphism $\bar{e}_F^n$ for all n.

These maps $\bar{e}_F^n$ (provided they exist) characterize in a certain sense forms in I^nF up to Witt equivalence. In this paper, however, we are interested in "structure" or "isometry" results about forms in I^nF. What we mean by results of this type is illustrated best by the following famous theorem which is due to Arason and Pfister [AP] and thus aptly named the "Arason-Pfister Hauptsatz" (we will refer to it later on simply by APH).

Arason-Pfister Hauptsatz *(Cf. [AP, Hauptsatz and Korollar 3].) Let* $0 \neq \varphi \in I^nF$ *be anisotropic. Then* $\dim \varphi \geq 2^n$. *If* $\dim \varphi = 2^n$ *then* $\varphi \in GP_nF$.

Before Merkuriev obtained his results on $\bar{e}_F^2$, and even before the above Hauptsatz, Pfister, in his attempt to classify forms in I^3F, proved the following (see also [S, Ch. 2, Theorem 14.4]).

Pfister's Theorem *(Cf. [P, Satz 14 and Zusatz].) Let* φ *be an even-dimensional form with* $\dim \varphi \leq 12$, $d_\pm \varphi = 1$ *and* $c(\varphi) = 1$. *Then* $\varphi \in I^3F$. *More precisely, the following holds.*

(i) *If* $\dim \varphi \leq 6$ *then* $\varphi = 0$ *in* WF, *i.e.* φ *is hyperbolic.*

(ii) *If* $\dim \varphi = 8$ *then* $\varphi \in GP_3F$.

(iii) *If* $\dim \varphi = 10$ *then* $\varphi \simeq \pi \perp \mathbb{H}$ *with* $\pi \in GP_3F$ *(here,* $\mathbb{H} \simeq \langle 1, -1 \rangle$ *denotes the hyperbolic plane). In particular,* φ *is isotropic.*

(iv) *If* $\dim \varphi = 12$ *then there exist* $r, s, t, u, v, w \in \dot{F}$ *such that*

$$\varphi \simeq r\langle\langle s \rangle\rangle \otimes \langle t, u, tu, -v, -w, -vw \rangle$$

(or, in WF, $\varphi = r(\langle\langle s, t, u \rangle\rangle - \langle\langle s, v, w \rangle\rangle)$*).*

Remark 1.1 (i) Pfister himself formulated part (iv) of the above theorem differently. It is not difficult to deduce the above statement from Pfister's original version. We will give an alternative proof of our version (and of part (iii) above) in Theorem 2.8.

(ii) Parts (i) and (ii) of the above theorem are essentially the Arason-Pfister Hauptsatz for $n = 3$.

Part (iii) of Pfister's theorem states that there are no anisotropic 10-dimensional forms in I^3F. This raises the question whether there is an analogous result for $n \geq 4$. Are all forms in I^nF of dimension $2^n + 2$ for $n \geq 4$ isotropic? One can go even further. For all $n \geq 2$ there exists a field F with an anisotropic form of dimension $2^n + 2^{n-1}$ in I^nF. This can easily be achieved "generically". Let $F = F_0(X_1, \cdots, X_{n-2}, Y_1, Y_2, Z_1, Z_2)$ be the rational function field in the variables X_i, Y_j, Z_k over an arbitrary field F_0, and let $\varphi \simeq \langle\langle X_1, \cdots, X_{n-2} \rangle\rangle \otimes \langle Y_1, Y_2, Y_1Y_2, -Z_1, -Z_2, -Z_1Z_2 \rangle$. In WF we have $\varphi = \langle\langle X_1, \cdots, X_{n-2}, Y_1, Y_2 \rangle\rangle - \langle\langle X_1, \cdots, X_{n-2}, Z_1, Z_2 \rangle\rangle$ and it follows that $\varphi \in I^nF$. The fact that φ is anisotropic follows readily from Springer's theorem (cf. [S, Ch. 6, Corollary 2.6]). Moreover, if $\varphi \in I^nF$ is anisotropic with $\dim \varphi > 2^n$ and $\varphi = \rho_1 + \rho_2$ in WF where $\rho_1, \rho_2 \in GP_nF$, then $\dim \varphi \geq 2^n + 2^{n-1}$ as follows from Lemma 2.3. These examples and facts together with Pfister's theorem lead quite naturally to the following conjecture (see also [Ka, Conjecture 9]).