A STOCHASTIC ANALOGUE OF THE VINOGRADOV MEAN VALUE THEOREM

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ABSTRACT. We prove a stochastic analogue of the well-known Vinogradov mean value theorem. In the special case of random variables taking values $1, 2, \ldots, P$ with equal probabilities our result yields an estimate of the same order in P as the original Vinogradov estimate.

1. Introduction and main results

Let $\mathcal{J}_m(P)$ denote the number of solutions of the simultaneous diophantine equations

$$\sum_{i=1}^{m}(x_i^j - y_i^j) = 0,\ j = 1, \ldots, k,$$

where $1 \leq x_i,\ y_i \leq P$ for $i = 1, \ldots, m$.
Since for any integer x we have

$$\int_0^1 \exp\{2\pi i\alpha x\}\, d\alpha = \begin{cases} 1, & \text{if } x = 0, \\ 0, & \text{if } x \neq 0, \end{cases}$$

we get

$$\mathcal{J}_m(P) = \int_0^1 \ldots \int_0^1 |\sum_{x=1}^{P} \exp\{2\pi i f(x)\}|^{2m} d\alpha_k \ldots d\alpha_1,$$

where $f(x) \triangleq \alpha_k x^k + \cdots + \alpha_1 x$.
Estimates of the order of magnitude of the growth of $\mathcal{J}_m(P)$ with increasing P have important applications in analytic number theory. The mean value $\mathcal{J}_m(P)$ was investigated by I.M.Vinogradov (1934).

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Theorem (I.M.Vinogradov). *Let $\tau > 0,\ k > 2$ be integers with $m \geq k\tau$ and let $P \geq 1$. Then*

$$\mathcal{J}_m(P) \leq C_\tau \cdot P^{2m-\triangle(\tau)},$$

where

$$\triangle(\tau) = 0.5k(k+1)(1-(1-1/k)^\tau),$$

$$C_\tau = (k\tau)^{6k\tau}\,(2k)^{4k(k+1)\tau}.$$

The bounds for $\mathcal{J}_m(P)$ provide information about the size of the trigonometric Weyl sums

$$F \triangleq \sum_{x=1}^{P} \exp\{2\pi i f(x)\}$$

and trigonometric sums involving functions to which there are reasonable polynomial approximations. Since there are numerous applications for such bounds Vinogradov was able to use his method with great success in many different problems of number theory. For later refinements of the Vinogradov mean value theorem see e.g. Stechkin (1975) and Wooley (1992), and the discussion and references therein.

It is obvious that if a random variable S takes values $1, 2, \ldots, P$ with the equal probability $1/P$ then

$$\mathbb{E}\exp\{2\pi i f(S)\} = F/P.$$

Moreover, using estimates for $|F|$ known in number theory it is possible to get bounds for $|\,\mathbb{E}\exp\{2\pi i f(S_n)\}|$, where S_n is a normed sum of independent identically distributed random variables $X_1, X_2, \ldots, X_n$ provided that a distribution of X_1 has a nondegenerate discrete component (see Götze et al. (1994)).

The aim of the present paper is to extend the Vinogradov mean value theorem to the case of general random variables. Although the stochastic extension is of independent interest, we intend to use these results to construct bounds for $|\,\mathbb{E}\exp\{2\pi i f(S_n)\}|$ without any assumption on the type of distribution of X_1. Our main result is the following

Theorem 1. *Let S denote a random variable, let $P \geq 1,\ k > 2,\ \tau \geq 1,\ m$ be positive integers. Put $f(x) = \alpha_k x^k + \cdots + \alpha_1 x$ and*

$$\mathcal{I}_m(P) \triangleq P^{-(k-1)k/2} \times \int_{-P^{k-1}}^{P^{k-1}} \int_{-P^{k-2}}^{P^{k-2}} \cdots \int_{-1}^{1} |\,\mathbb{E}\exp\{2\pi i f(S)\}\cdot \mathbf{1}_{\{-P<S\leq P\}}|^{2m} d\alpha_k \ldots d\alpha_1, \tag{0-1}$$

where $\mathbf{1}_{\{A\}}$ denotes an indicator function of an event A.

Then for all $m = k\tau$ there exists a constant $D_\tau = D(k,\tau)$ depending on k and τ only such that

$$\mathcal{I}_m(P) \leq D_\tau \cdot P^{2m-\triangle(\tau)} \left(\sup_I \mathbb{P}\,(S \in I)\right)^{2m}, \tag{0-2}$$