OPTIMAL RATES OF CONVERGENCE IN FUNCTIONAL LIMIT THEOREMS FOR QUADRATIC FORMS

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Abstract. Let $X, X_1, X_2, \dots$ be a sequence of i.i.d. random vectors taking values in a k-dimensional real linear space $\mathbf{R}^k$. Assume that $\mathbf{E}\, X = 0$ and that X is not concentrated in a proper subspace of $\mathbf{R}^k$. Let G denote a mean zero Gaussian random variable with the same covariance operator as that of X. We investigate the distributions of non-degenerated quadratic forms $Q[S_N]$ of the normalized sums $S_N = N^{-1/2}\,(X_1 + \dots + X_N)$ and show that

$$\sup_x \Big|\, \mathbf{P}\,\{\, Q[S_N] \le x \,\} - \,\mathbf{P}\,\{\, Q[G] \le x \,\}\,\Big| = \mathcal{O}\big(N^{-1}\big),$$

provided that $k \ge 9$ and the fourth moment of X exists. The bound $\mathcal{O}\big(N^{-1}\big)$ is optimal and improves, e.g., the well-known bound $\mathcal{O}\big(N^{-k/(k+1)}\big)$ due to Esseen (1945). The result extends to the case of random vectors taking values in a Hilbert space.

1. Introduction

Let $\mathbf{R}^k$ denote the k-dimensional space of real vectors $x = (x_1, \dots, x_k)$ with scalar product $\langle \cdot, \cdot \rangle$ and norm $|x|^2 = \langle x, x \rangle = x_1^2 + \dots + x_k^2$. Since our results are independent of the dimension (provided $k \ge 9$), it will be convenient to denote by $\mathbf{R}^\infty$ a real separable Hilbert space. Thus, $\mathbf{R}^\infty$ consists of all real sequences $x = (x_1, x_2, \dots)$ such that $|x|^2 = x_1^2 + x_2^2 + \dots < \infty$.

Let $X, X_1, X_2, \dots$ be a sequence of i.i.d. random variables taking values in $\mathbf{R}^k$. If $\mathbf{E}\, X = 0$ and $\mathbf{E}\,|X|^2 < \infty$, then the sums

$$S_N = N^{-1/2}\,(X_1 + \dots + X_N)$$

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converge weakly to a mean zero Gaussian random variable, say G, such that its covariance is equal to the covariance of X. We shall assume that G (or, equivalently, X) is not concentrated on a subspace of $\mathbf{R}^k$, i.e., that $\mathbf{P}\{G \in \mathbf{L}\} < 1$ whenever $\mathbf{L} \subset \mathbf{R}^k$ is a proper closed linear subspace of $\mathbf{R}^k$.

For a linear symmetric and bounded operator $Q : x \mapsto Qx$ mapping $\mathbf{R}^k$ into $\mathbf{R}^k$, define the quadratic form $Q[x] = \langle Qx, x\rangle$. Assume that the operator Q is non-degenerated, i.e., that $\ker Q = \{0\}$.

Notice that the distribution of the quadratic form $Q[G]$ may be represented up to a shift as the distribution of a finite (resp., infinite) weighted sum of squares of i.i.d. standard Gaussian variables, for $k < \infty$ (resp., $k = \infty$).

Theorem 1.1. *Let* $\mathbf{E}\,X = 0$ *and* $\beta_4 = \mathbf{E}\,|X|^4 < \infty$. *Assume that* $k \geq 9$ *or* $k = \infty$. *Then*

$$\sup_x \Big|\, \mathbf{P}\{Q[S_N] \leq x\} - \mathbf{P}\{Q[G] \leq x\}\,\Big| = \mathcal{O}\big(N^{-1}\big).$$

The constant in this bound depends on β_4, *the eigenvalues of* Q *and the covariance operator of* G *only.*

Remark. A rather straightforward inspection of the proofs shows that Theorem 1.1 holds for $k = 8$ with $\mathcal{O}(N^{-1}\ln^{\delta} N)$ instead of $\mathcal{O}(N^{-1})$, for some $\delta > 0$.

For an orthonormal system of vectors $e_1, \ldots, e_s \in \mathbf{R}^k$ and a random variable $X \in \mathbf{R}^k$, consider the condition

$$\mathbf{P}\Big\{\,|\widetilde{X} - e_j| \leq (100\,s)^{-1}\Big\} \geq p, \qquad \mathbf{P}\Big\{\,|\widetilde{X} - Qe_j| \leq (100\,s)^{-1}\Big\} \geq p, \tag{1.1}$$

for all $1 \leq j \leq s$, where $\widetilde{X}$ denotes the symmetrization of X, i.e., $\widetilde{X}$ has the distribution $\mathcal{L}(\widetilde{X}) = \mathcal{L}(X_1 - X_2)$.

Introduce the concentration function

$$Q(X;\,\lambda) = Q(X;\,\lambda;\,Q) = \sup_{a,\,x} \mathbf{P}\big\{\,x \leq Q[X + a] \leq x + \lambda\,\big\}, \quad \text{for} \quad \lambda \geq 0.$$

Theorem 1.2. *Assume that the operator* Q *is isometric. If* $s \geq 9$ *and* X *satisfies* (1.1) *with some* $p > 0$, *then, for sums* $Z_N = X_1 + \cdots + X_N$,

$$Q(Z_N;\,\lambda) \leq \frac{c_s}{p\,N}\,\max\big\{1;\,\lambda\,\big\}. \tag{1.2}$$

The bound (1.2) may be useful for investigations of infinitely divisible approximations. Notice that the bound (1.2) is an analogue for $Q[Z_N]$ of the classical bound $\mathcal{O}\big(N^{-1/2}\max\{1;\,\lambda\}\big)$ for the concentration of sums (see, e.g., Th. 9 of Ch. 3 in Petrov (1975)). Furthermore, the assumption that Q is isometric, i.e., that $Q^2 = I$, does not restrict the generality. Indeed, any symmetric operator Q