SUBGROUP GROWTH OF FREE PRODUCTS

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Introduction

For a group Γ denote by $s_\Gamma(n)$ the number of subgroups of index n in Γ. If Γ is finitely generated (as we will always suppose) or of finite (subgroup) rank then $s_\Gamma(n)$ is finite for all n. The concept of *subgroup growth*, which has evolved during the last decade, provides a framework for investigations concerning the arithmetic properties of the sequence $\{s_\Gamma(n)\}_1^\infty$ or related subgroup counting functions and their connection with the algebraic structure of the group Γ. Like Milnor's concept of word growth it can be thought of as a chapter in "asymptotic group theory", where one studies groups from the point of view of the asymptotic behaviour of some of their invariants. Some of the major developments concerning the function $s_\Gamma(n)$ are described in Lubotzky's Galway notes [L1]; cf. also [GSS], [L2], [LM], [LMS], [dS] and the references therein.

The purpose of the present paper is to investigate the growth behaviour and the asymptotics of $s_\Gamma(n)$ in the case when Γ is a free product of the form

$$\Gamma = \mathop{*}_{\sigma=1}^{s} G_\sigma \, * \, F_r, \quad 0 \leq r,\ s < \infty, \quad 1 < m_\sigma = |G_\sigma| < \infty. \tag{1}$$

Here, the G_σ are arbitrary finite groups and F_r denotes the free group of rank r. Our main result (Theorem 1) gives a complete asymptotic expansion for $s_\Gamma(n)$ in terms of the numbers r, s and the local data m_σ and $s_{G_\sigma}(d_\sigma)$ for $\sigma = 1, \ldots, s$ and $d_\sigma | m_\sigma$. We also consider the question as to what kind of information on the decomposition (1) of such a group Γ is contained in the function $s_\Gamma(n)$ or its asymptotics, and we show that this combinatorial information determines Γ up to finitely many isomorphism types; cf. Theorem 3. Our results depend on the recent solution in [M5] of another problem from "asymptotic group theory" concerning finite group actions:

(2) *Given a finite group* G, *how does the number of finite group actions* ${}_GX$ *behave as* $|X| \to \infty$?

This problem is of considerable interest in its own right; those results on (2) needed in our context are presented in §1 together with some historical comments. For proofs and more details on the history of (2) the reader is referred to [M5].