Splitting property in infinite posets

Péter L. Erdős[1,2]
Department of Computer Science
University of Agricultural Science, Gödöllő, Hungary

December 7, 1995

AMS Subject Classifications (1991): 05 D 05, 06 A 07

Key words: maximal antichain, infinite poset

Abstract

It is known that in every finite poset $\mathcal{P}$ any maximal antichain S with some *denseness* property may be partitioned into disjoint subsets S_1 and S_2, such that the union of the downset of S_1 with the upset of S_2 yields the entire poset: $\mathcal{D}(S_1) \cup \mathcal{U}(S_2) = \mathcal{P}$. Hereby we give analogues results for infinite posets.

1 Introduction

Let $\mathcal{P} = (P, <_P)$ be an arbitrary (finite or infinite) partially ordered set (poset) and let H be a subset of P. The *downset* $\mathcal{D}(H)$ of the subset H is:

$$\mathcal{D}(H) = \{x \in P : \exists s \in H (x \leq s)\}.$$

The *upset* of H is:

$$\mathcal{U}(H) = \{x \in P : \exists s \in H (s \leq x)\}.$$

[1]Research was partially carried out on a visit to Bielefeld, Germany
[2]Research was partially supported by OTKA under grant no. T 016358

We introduce also the sets

$$\mathcal{D}^*(H) = \{x \in P : \exists s \in H(x < s)\}$$

and

$$\mathcal{U}^*(H) = \{x \in P : \exists s \in H(s < x)\}.$$

A subset $S \subset P$ is called *antichain* or *Sperner system* if no two elements of S are comparable. An antichain S is *maximal* if for every antichain $S' \subset P$, $S \subset S'$ implies $S = S'$. It is easy to see that S is a maximal antichain iff

$$\mathcal{D}(S) \cup \mathcal{U}(S) = P. \tag{1}$$

Definition A subset $H \subset P$ is called a *generator* if it satisfies condition (1) (S replaced by H.). We say that a generator H satisfies the *splitting property* if there exists a partition (H_1, H_2) of H into disjoint subsets such that

$$\mathcal{U}(H_1) \cup \mathcal{D}(H_2) = P \tag{2}$$

holds. Finaly a subset $H \subset P$ is called *dense* in the poset $\mathcal{P}$ if there is no *open interval* $\langle x, y \rangle = \{z \in P : x < z < y\}$) which intersects H in exactly one element.

In [1] R. Ahlswede, P.L. Erdős and N. Graham have shown that

Theorem 1 *Let S be a maximal, dense antichain in the finite poset $\mathcal{P}$. Then S satisfies the splitting property.*

In fact, there are a lot of finite posets where every maximal antichain satisfies the splitting property. One example for this is the Boolean algebra of n elements. On the other hand there was already in paper [1] posed the question: what can someone say about the infinite case. The first results in that direction were proved by R. Ahlswede and L. Khachatrian in [2]. In this paper we prove several results on splitting properties of infinite posets.

2 Results

In this paper a partially ordered set (poset) means always an arbitrary, finite or infinite one, except it is stated otherwise.