Invariant Manifolds for Non-autonomous Systems with Application to One-step Methods

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Abstract

In this paper we study the existence of invariant manifolds for a special type of nonautonomous systems which arise in the study of discretization methods. According to [10], a one-step scheme of step-size ε for an autonomous system can be interpreted as the ε-flow of a perturbed nonautonomous system. The perturbation is 'rapidly forced' in the sense that it is periodic with respect to time with period ε. Assuming a saddle node for the autonomous system, we prove that these rapidly forced perturbations have center manifolds which exist in a uniform neighborhood and which converge to a center manifold of the autonomous system as ε tends to zero. Our results are applied to obtain a smooth continuation as well as estimates of the well known center manifolds for one-step schemes. They also form the basis for studying saddle-node homoclinic orbits under discretization.

Keywords: Invariant manifolds, center manifolds, nonautonomous systems, one-step methods, centered Euler scheme.
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1 Introduction

In recent years many efforts have been directed towards a deeper understanding of the effects caused by discretizing a continuous dynamical system, see [13], [20] for recent reviews.

Consider a smooth dynamical system

$$\dot{x} = f(x), \; x \in \mathbb{R}^m, \tag{1.1}$$

and let $F(t, x)$ be the induced t-flow. We are interested in the longtime behavior of one-step methods

$$x_{n+1} = \Phi(\varepsilon, x_n), \; n = 0, 1, 2, \cdots \tag{1.2}$$

Here ε denotes a constant but small step size and $\Phi(\varepsilon, \cdot)$ is a smooth mapping in $\mathbb{R}^n$ that approximates the ε-flow $F(\varepsilon, \cdot)$ to a certain order

$$|F(\varepsilon, x) - \Phi(\varepsilon, x)| \leq C\varepsilon^{p+1}, \quad \forall x \in \mathbb{R}^m. \tag{1.3}$$

For estimates of this type we refer to [5], [11], [20].

For certain structurally stable systems (1.1), it is possible to set up an ε-dependent topological conjugacy between the maps $F(\varepsilon, \cdot)$ and $\Phi(\varepsilon, \cdot)$ and to estimate its distance to the identity in powers of ε (see [12], [13], [14]). In general such an embedding is impossible and some of the obstructions when periodic orbits are present are discussed in [12], [13], [14].

An important structurally unstable situation occurs if the system (1.1) has a saddle with a homoclinic orbit. This situation was analyzed in detail in [10] for a parametrized family. The basis assumption in [10] is that the homoclinic orbit, as a function of time, has an analytic extension to a strip in the complex plane. Then it is shown that generically the map $\Phi(\varepsilon, \cdot)$ has transversal homoclinic points and that the angle of intersection of the stable and unstable manifolds is exponentially small, that is of the order $O(e^{-c/\varepsilon})$.

The basic tool in [10] is to view the one step map $\Phi(\varepsilon, \cdot)$ of order p as the 0 to ε flow of a perturbed nonautonomous system

$$\dot{x} = f(x) + \varepsilon^p g(\varepsilon, t/\varepsilon, x) \tag{1.4}$$

where g is smooth and 1-periodic in its second variable. The perturbation term in (1.4) then has period ε in t and such perturbations are called **rapidly forced** in [10].