ON THE DOMAIN OF ATTRACTION TO THE NORMAL LAW AND CONVERGENCE OF SELFNORMALIZED SUMS

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Abstract

It is proved that if the selfnormalized sums for i.i.d. random variables with the law of X are asymptotically standard normal, and if there is, additionally, convergence of some moments, then X is in the domain of attraction of the normal law. It is also proved that if X is in the domain of attraction of some α–stable law, $\mathbf{E}X=0$ for $1<\alpha\leq 2$, X symmetric for $\alpha=1$, then all the moments of the selfnormalized sums converge, in particular showing that the previous result has a converse. A characterization of the domain of attraction of the normal law in terms of the asymptotic neglibility of the maximum term of the sum with respect to the sum centered at its median is also presented.

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1. Introduction. Let $X, X_i,\ i \in \mathbf{N}$, be independent, identically distributed random variables and let

$$S_n = \sum_{i=1}^{n} X_i, \quad V_n^2 = \sum_{i=1}^{n} X_i^2, \quad n \in \mathbf{N}.$$

Logan et *al.* conjecture that 'S_n/V_n is asymptotically normal if [and perhaps only if] X is in the domain of attraction of the normal law' (and X is centered). The 'if' part of this conjecture follows rather easily from basic principles (Raikov's theorem) but the parenthetical 'only if' is unknown in the non–symmetric case. Griffin and Mason (1991) attribute to Roy Erickson a beautiful proof of the fact that, for X *symmetric*, convergence in distribution of the selfnormalized sums S_n/V_n to the standard normal law implies that X is in the domain of attraction of the normal law ($X \in DAN$). The proof relies heavily on symmetry, and the non–symmetric case seems to be, perhaps surprisingly, quite harder. The present paper is devoted to this problem.

Our main result states that, for X integrable and centered, if $\mathcal{L}(S_n/V_n) \to_w N(0,1)$ and $\sup_n \mathbf{E}(S_n/V_n)^6 < \infty$, then X is indeed in the domain of attraction of the normal law (uniform boundedness of the sixth moment can be replaced by some mild regularity on the law of X). We should remark that this constitutes a characterization of the domain of attraction of the normal law because we also prove (and this is our second main result) that, if X is in the domain of attration of any α–stable law, assuming $\mathbf{E}X = 0$ for $1 < \alpha \leq 2$ and X symmetric for $\alpha = 1$, then *all* the moments of the selfnormalized sums converge. This is another aspect of the general fact that *selfnormalization improves convergence in the CLT*. Perhaps the most important instance of this fact occurs in the paper by Bentkus and Götze (1994) where they show that this is the case for the speed of convergence of the cdf of selfnormalized sums to that of the standard normal distribution when $X \in DAN$.

When we started this research we were unaware of O'Brien's result to the effect that $X \in DAN$ if and only if $\max_{i \leq n} |X_i|/V_n \to 0$ in probability, a fact generally known in the symmetric case (e.g., Griffin and Mason, 1991), that we needed for the proof of the main result. We independently obtained a proof of this fact, as well as of the following related result, which seems of independent interest: $X \in DAN$ if and only if $\max_{i \leq n} |X_i|/|S_n - m_n| \to 0$ in probability, where m_n denotes a median of S_n. That is, normal convergence is characterized by the maximum term being asymptotically negligible in probability with respect to the sum *centered at its median* (any of its medians). This paper also contains two other, perhaps less interesting, characterizations of the domain of attaction of the normal law.

In order to treat S_n/V_n we must deal with the quantities $\Lambda_m := (n!/(n-m)!)\mathbf{E}\big[X_1 \cdots X_m/V_n\big]$. To this effect we use the simple fact that $\int_0^\infty \lambda^{r-1} \exp(-\lambda a) d\lambda = (r-1)!/a^r$ ($r \in \mathbf{N}$ and $a > 0$) to express the quantities Λ_m as averages of products of expected values of the form $\mathbf{E}f(X)$. In a sense, this works as a de Finetti representation for the *finite* family of exchangeable variables $X_1/V_n, \ldots, X_n/V_n$. This allows, in addition, to obtain a Hölder type inequality, $\Lambda_4 \leq \Lambda_2^{1/2}\Lambda_6^{1/2}$, which is crucial in the proof of the main result (thus the sixth moment!). These are the two main new ingredients in our proofs.

Obviously, the problem of characterizing the domain of attraction of the normal law by the CLT for selfnormalized sums is interesting in its own right. In fact, some of our