ON SHIFTS OF CONVEX MEASURES

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According to a classical theorem, for every Gaussian measure γ on a locally convex space X and every vector $h \in X$, the measures γ and $\gamma(\,\cdot\,+h)$ are either equivalent or mutually singular. This property has a lot of applications in both the theory of random processes and infinite dimensional analysis. The collection of all vectors h, for which such measures are equivalent is said to be the Cameron – Martin space of γ. It is known that for every element h of this space the mapping $t \mapsto \gamma(\,\cdot\,+th)$ is infinitely differentiable. This fact lies in the foundations of the analysis on the Wiener space (see [4]). Naturally the question arises about other classes of measures for which similar assertions are true.

In view of diverse applications in the stochastic analysis (see [1], [2]), it is of interest to study the differential properties of convex measures. A Gaussian measure is a very special case of a convex measure. In this paper we study the behavior of convex Radon measures on a locally convex topological vector space under shifts. One of the main results we obtain is the estimate

$$\|\mu_h - \mu\| \geq 2 - 2e^{\frac{-\|d_h\mu\|}{2}},$$

for the finite dimensional projections of the measure μ, where $d_h\mu$ denotes the Skorohod derivative of μ along h. Making use of this estimate, we prove that for every direction h, which is not a direction of differentiability of the measure μ in the sense of Skorohod, the measures μ and μ_h are mutually singular. Therefore, the above mentioned dichotomy extends from Gaussian measures to convex ones in the following form: a convex measure is either differentiable along a vector in the sense of Skorohod or all its translates in this direction are mutually singular.

Let us recall some definitions. Let X be a locally convex topological vector space, and $\mathcal{B}(X)$ its Borel σ-field. The term "measure on the space X" means a Radon (possibly, signed) measure on $\mathcal{B}(X)$ (see, e.g., [10]). For any measure μ on X we

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denote by $|\mu|$ its variation and by $\|\mu\| := |\mu|(X)$ its variation norm. Continuous linear mappings $P : X \to R^n$ are called finite dimensional projections. Sets of the form $C = P^{-1}(B)$, where $P : X \to R^n$ is a finite dimensional projection and $B \in \mathcal{B}(R^n)$, are said to be cylindrical. The induced measure $\mu \circ P^{-1}$ on R^n is defined by $\mu \circ P^{-1}(B) = \mu(P^{-1}(B))$. Finally, put $\mu_h(B) = \mu(B + h)$.

A measure μ on X is said to be Skorohod differentiable along a vector h if there exists a (signed) Radon measure ν such that for all continuous bounded functions f on X the following equality holds true:

$$\lim_{t \to 0} \int \frac{f(x - th) - f(x)}{t} \, \mu(dx) = \int f(x) \, \nu(dx).$$

Such a measure ν (which is clearly unique) is denoted by $d_h\mu$ and called the Skorohod derivative of μ along h (see [3], [5] for more details on differentiable measures).

In the case $X = R^n$, a measure μ is Skorohod differentiable along a vector h if and only if its partial derivative $\partial_h\mu$ in the sense of generalized functions (see [9], [11]) is a bounded measure. A measure on R^n is Skorohod differentiable along n independed vectors if and only if it is of the form $\mu = f\,dx$, where f is an integrable function of the class $BV(R^n)$ (see [11]).

A probability measure μ on X is said to be convex (see [6]) if

$$\mu(\lambda A + (1 - \lambda)B) \geq \mu(A)^\lambda \mu(B)^{1-\lambda},$$

for all sets A and B from the σ-field $\sigma(X)$, generated by the cylindrical sets (i.e., by the dual X^*).

This condition is equivalent to

$$\mu_*(\lambda A + (1 - \lambda)B) \geq \mu(A)^\lambda \mu(B)^{1-\lambda},$$

for all A, $B \in \mathcal{B}(X)$, $\lambda \in [0, 1]$, where $\mu_*(A) = \sup\{\mu(K),\ K \subset A, K \text{ is compact}\}$. Clearly, in the case where X is a Banach (or Fréchet) space, the set $\lambda A + (1 - \lambda)B$ is μ-measurable (since μ has separable support), so that one can replace μ_* by μ.

It was proved in [6] that a measure μ is convex if and only if all its finite dimensional projections are convex. In the same work the following theorem was proved, describing all convex measures on finite dimensional spaces.

Theorem 0. *A probability measure μ on R^n is convex if and only if there exist an integer number k, $0 \leq k \leq n$, a probability measure ν on R^k, absolutely continuous with respect to the Lebesgue measure m_k, and an affine transformation $h : R^k \to R^n$ such that $\mu = \nu \circ h^{-1}$ and the function $-\log(d\nu/dm_k)$ is convex if $k > 0$ (where $\log 0 = -\infty$).*

In particular, any convex measure on R^n is concentrated on an affine subspace of R^n on which it has a density with respect to the corresponding Lebesgue measure.

According to [8], every absolutely continuous convex measure on R^n is Skorohod differentiable along all vectors. In particular, every such measure has a density of