ON MACLANE COHOMOLOGY FOR THE RING OF INTEGERS

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1. Introduction

For a ring R, let $\mathcal{F}(R)$ be the category whose objects are the functors from the category of finitely generated free left R-modules to the category of left R-modules, and whose maps are the natural transformations; one such functor is the inclusion functor I. The MacLane cohomology of the ring R with coefficient in a functor T of $\mathcal{F}(R)$ is defined by

$$\mathrm{HML}^*(R,T) = \mathrm{Ext}^*_{\mathcal{F}(R)}(I,T) \ .$$

When T is just tensoring with a R-R-bimodule M, we write $\mathrm{HML}^*(R,M)$ for $\mathrm{HML}^*(R, -\otimes_R M)$. By [8], this is compatible with MacLane's original definition [10].

There is a dual notion, MacLane homology, known [6, 12] to be isomorphic to the topological Hochschild homology in the sense of Bökstedt [2] and stable K-theory in the sense of Waldhausen [9]. Using sophisticated topological methods Bökstedt has calculated $\mathrm{THH}(\mathbb{Z})$ and $\mathrm{THH}(\mathbb{Z}/p\mathbb{Z})$, for all primes p [3]. Bökstedt's result for $\mathbb{Z}/p\mathbb{Z}$ is closely related to L. Breen's calculations [4]. Since, V. Franjou, J. Lannes and L. Schwartz [7] have given algebraic computations of $\mathrm{HML}^*(\mathbb{Z},\mathbb{Z}/p\mathbb{Z})$ and $\mathrm{HML}^*(\mathbb{Z}/p\mathbb{Z}, Sym^k)$, where Sym^k denotes the k-th symmetric power; their results include the module structure over $\mathrm{HML}^*(\mathbb{Z}/p\mathbb{Z},\mathbb{Z}/p\mathbb{Z})$ defined by the Yoneda product.

The purpose of this paper is to give a short and elementary algebraic calculation of $\mathrm{HML}^*(\mathbb{Z},\mathbb{Z})$ which includes the Yoneda ring structure. We prove:

Theorem. *Let $\Gamma(x)$ be the free divided-powers algebra on a generator x of degree 2. The algebra $\mathrm{HML}^*(\mathbb{Z},\mathbb{Z})$ is isomorphic to the quotient of $\Gamma(x)$ by its ideal generated by the class x.*

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This theorem is proved in section 4, using a spectral sequence whose first line is the desired $\mathrm{HML}^*(\mathbb{Z},\mathbb{Z})$, and which converges to zero. This spectral sequence is constructed in section 3. Its E_2-term (but the first line) is computed by the corollary 2.2.

2. MacLane cohomology with coefficients in symmetric or exterior powers

For an abelian group A, we denote by Sym^*A the symmetric algebra generated by A. Our first task is the computation of $\mathrm{HML}^*(\mathbb{Z}, Sym^n)$ as a right module over $\mathrm{HML}^*(\mathbb{Z},\mathbb{Z}/p\mathbb{Z})$ for each n but for $n=1$.

For a given prime p, the rings maps:

$$\mathrm{HML}^*(\mathbb{Z},\mathbb{Z}) \to \mathrm{HML}^*(\mathbb{Z};\mathbb{Z}/p\mathbb{Z}) \leftarrow \mathrm{HML}^*(\mathbb{Z}/p\mathbb{Z},\mathbb{Z}/p\mathbb{Z})$$

will be used to relate the module structures over these different rings. Let us recall from [7, par. 9] the computation of the right-hand side map.

We let Λ denote the following graded algebra:

$$\Lambda := \mathbb{Z}/p\mathbb{Z}[e_0, ..., e_h, ...]/({e_h}^p; h \geq 0) \ ,$$

where e_h is a class of degree $2p^h$. Let Λ_k be the factoralgebra of Λ by the ideal generated by $e_0, ..., e_{k-1}$. There exist isomorphisms of graded algebras:

$$\begin{aligned} \mathrm{HML}^*(\mathbb{Z}/p\mathbb{Z},\mathbb{Z}/p\mathbb{Z}) &\cong \Lambda && [7, \text{Théorème } 7.3], \\ \mathrm{HML}^*(\mathbb{Z},\mathbb{Z}/p\mathbb{Z}) &\cong \Lambda_1 \otimes \Lambda(\xi_1) && [7, \text{par. } 9.2], \end{aligned}$$

where ξ_1 is a class of degree $2p-1$ and $\Lambda(\xi)$ denotes the exterior algebra on a generator ξ. Under these isomorphisms, the map

$$\mathrm{HML}^*(\mathbb{Z}/p\mathbb{Z},\mathbb{Z}/p\mathbb{Z}) \to \mathrm{HML}^*(\mathbb{Z};\mathbb{Z}/p\mathbb{Z})$$

is the composite: $\Lambda \twoheadrightarrow \Lambda_1 \hookrightarrow \Lambda_1 \otimes \Lambda(\xi_1)$.

Proposition 2.1. *Let n be an integer, $n \geq 2$.*

If n is not a prime power, then: $\mathrm{HML}^*(\mathbb{Z}, Sym^n) = 0$.

If $n = p^h$ is a power of a prime p, the right module structure over $\mathrm{HML}^*(\mathbb{Z},\mathbb{Z})$ *factors through* $\mathrm{HML}^*(\mathbb{Z};\mathbb{Z}/p\mathbb{Z})$, *making* $\mathrm{HML}^*(\mathbb{Z}, Sym^n)$ *a free Λ_h-module on a class of degree* 1. *In particular:*

$$\mathrm{HML}^i(\mathbb{Z}, Sym^n) = \begin{cases} \mathbb{Z}/p\mathbb{Z} & \text{if } \ i \equiv 1 \pmod{2n} \\ 0 & \text{otherwise.} \end{cases}$$

Proof. By [11] (or by [7, lemme 0.4], see also the appendix of [1]) we know that $\mathrm{HML}^*(\mathbb{Z}, Sym^i \otimes Sym^j)$ is 0, for $i,\ j \geq 1$. Thus by [5, 10.9], $\mathrm{HML}^*(\mathbb{Z}, Sym^n)$ is annihilited by the binomial coefficient $\binom{n}{i}$, for $1 \leq i \leq n-1$.

By [5, 10.10] we first get: $\mathrm{HML}^*(\mathbb{Z}, Sym^n) = 0$ if $n \geq 2$ is not a prime power; now fixing a prime p and setting $n = p^k \geq 2$ we get as well that $\mathrm{HML}^*(\mathbb{Z}, Sym^n)$ is annihilited by p.