THE REDUCTION MAP FOR THE ÉTALE K-THEORY OF A CURVE

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Abstract. In the present work, we investigate the reduction map on the étale K-theory of a curve defined over a global field. We prove that on the even-dimensional K-groups the map has finite kernel and reduce the odd-dimensional case to a conjecture of Jannsen.

1. Introduction

The algebraic K-groups of arithmetic schemes are expected to carry deep arithmetic and geometric information, according to Beilinson's striking conjectures which relate them to special values of L-functions. Yet still little is known on their structure. In the case of the ring of integers $\mathcal{O}_K$ of a number field K, Quillen proved that the groups $K_i(\mathcal{O}_K)$ are finitely generated and Borel computed their ranks. In [AB] D.Arlettaz and the first author investigated the kernel of the natural map

$$K_i(\mathcal{O}_K) \to \prod_{\mathfrak{p} \subseteq \mathcal{O}_K} K_i(\kappa_{\mathfrak{p}}) \tag{1.1}$$

where $\mathfrak{p}$ runs through the maximal ideals of $\mathcal{O}_K$, and proved that this kernel is finite. This is trivial for i even, since then $K_i(\mathcal{O}_K)$ itself is finite; the difficulty is for i odd. We note that the same proof yields the same result for rings of integers of global fields of positive characteristic.

In this paper, we consider the next-dimensional case, that of a curve X over a global field K. Fix an odd prime number l different from the characteristic of K. Let S be a finite set of places of K containing the places above l and the places where X has bad reduction, and let $\mathcal{O}_S$ denote the ring of S-integers in K. Let $\mathcal{X}$ be a smooth, proper model of X over $Spec\, \mathcal{O}_S$. Let $K_i^{ét}(\mathcal{X})$ denote the i-th l-adic étale K-group of $\mathcal{X}$, in the sense of Dwyer-Friedlander [DF]. Our main result is then:

Theorem. *For a finite place $v \notin S$, let $\mathcal{X}_v$ be the fibre of $\mathcal{X}$ above v. Then the kernel of the reduction map*

$$K_i^{ét}(\mathcal{X}) \to \prod_{v \notin S} K_i^{ét}(\mathcal{X}_v)$$

is finite for i even.

Notice the shift from odd to even from the case of rings of integers to that of curves. We expect this theorem to hold when replacing étale K-groups by ordinary

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K-groups, but have too little information on the former to be able to do that. In [AB], this was made possible by the Soulé-Dwyer-Friedlander theorem that the maps $K_i(\mathcal{O}_K) \to K_i^{ét}(\mathcal{O}_K)$ have finite kernel; however, nothing is known on this kernel in the case of $\mathcal{X}$.

The proof of the theorem is not very difficult. We prove a somewhat finer result, namely that an element of infinite order in $K_i^{ét}(\mathcal{X})$ goes by the reduction map to a collection of elements of arbitrary large orders in the factors $K_i^{ét}(\mathcal{X}_v)$ (this implies the theorem because $K_i^{ét}(\mathcal{X})$ is a finitely generated $\mathbb{Z}_l$-module). To do this, we reduce through the Dwyer-Friedlander spectral sequence (relating the étale K-groups of $\mathcal{X}$ to its continuous étale cohomology) and the Leray spectral sequence for the morphism $\mathcal{X} \to Spec\,\mathcal{O}_S$ (relating the latter to S-ramified Galois cohomology of K) to a similar statement which replaces the étale K-groups by $H^1(G_S, T_l(J)(n))$ and $H^1(\kappa_v, T_l(J)(n))$. Here G_S denotes Galois group of the maximal S-ramified extenison of K, J is the Jacobian variety of $\overline{X} = X \times_K \overline{K}$, $T_l(J)$ is its Tate module and n is a Tate twist. The result then follows from a result of Serre [Se] (compare [J1]) followed by a Chebotarev density argument, as in [AB].

This paper is organized as follows. In section 2 we collect étale K-theory computations using the two spectral sequences mentioned above. In section 3, we prove the reduction result on $H^1(G(\bar{K}/K), T_l(A)(k))$ for any abelian variety A defined over K. In section 4 we prove the theorem. Finally, in section 5, we discuss the case of odd-dimensional étale K-groups.

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Convention. In all this paper, l is an *odd* prime.

2. Cohomological and the Étale K-theoretic Computation

In this section we collect some facts about the continuous étale cohomology groups of $\mathcal{X}$ with values in the l-adic sheaf $\mathbb{Z}_l(k)$ as well as its l-adic étale K-groups. We briefly recall the definition of the former groups, [J2, p. 216]. Let $(\mathcal{F}_m)$ be a projective system of $\mathbb{Z}/l^m$-sheaves on $\mathcal{X}$. The functor which sends $(\mathcal{F}_m)$ to $\varprojlim H^0(\mathcal{X}, (\mathcal{F}_m))$ is left exact and we define $H^i_{cts}(\mathcal{X}, (\mathcal{F}_m))$ to be its ith right derived functor. When the projective system is $\mathbb{Z}_l(k)$, we denote these groups by $H^i_{cts}(\mathcal{X}, \mathbb{Z}_l(k))$. We will need the following properties of continous cohomology.

(a) There is an exact sequence [J2, p.216, (3.1)]:

$$(2.1) \qquad 0 \to \varprojlim{}^1 H^{i-1}(\mathcal{X}, \mathcal{F}_m) \to H^i_{cts}(\mathcal{X}, (\mathcal{F}_m)) \to \varprojlim H^i(\mathcal{X}, \mathcal{F}_m) \to 0.$$

In particular, if $H^i(\mathcal{X}, \mathcal{F}_m)$ are finite for any i and m, then by the Mittag-Leffler condition $H^i_{cts}(\mathcal{X}, (\mathcal{F}_m)) \cong \varprojlim H^i(\mathcal{X}, \mathcal{F}_m)$.

(b) Let $f : \mathcal{X} \to Spec\,\mathcal{O}_S$ be the natural map. Then there is a spectral sequence [J2, p. 219, (3.10)]:

$$(2.2) \qquad E_2^{r,s} = H^r_{cts}(\mathcal{O}_S, R^s f_*(\mathcal{F}_m)) \Rightarrow H^{r+s}_{cts}(\mathcal{X}, (\mathcal{F}_m))$$