Regular two-graphs from the even unimodular lattice $E_8 \oplus E_8$

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Abstract

This is a revised version of the first part of [8]. Each two-graph is in one-to-one correspondence with a set of equiangular lines. This implies that a two-graph is represented by a system of vectors of equal odd norm with mutual inner products ± 1. This is a special odd system, i.e. a set of vectors with odd inner products. The construction of obtaining odd systems from doubly even lattices introduced in [5], [6] is applied to the even unimodular lattice $E_8 \oplus E_8$ multiplied by $\sqrt{2}$. For the odd system of 456 vectors obtained by this construction, we study odd subsystems of 36 vectors spanning equiangular lines, i.e. subsystems representing regular two-graphs on 36 points. A subsystem of vectors representing a two-graph generates a sublattice of the lattice $E_8 \oplus E_8$. These sublattices are distinguished by sets of lattice vectors of norm 2. These sets are root systems. Hence the set of all two-graphs from $E_8 \oplus E_8$ is partitioned into families of two-graphs with the same root system. It is proved here that there are 7 families with the following root systems: $\emptyset$, A_1^7, $A_2A_3^3$, A_6A_7, $A_1^9D_4$, $A_1A_3D_4D_6$, D_7E_7. The two-graphs with the 4 first root systems relate to the 23 Steiner triple systems with a head. We prove that the 100 first two-graphs of the list of two-graphs obtained by E.Spence by use a computer [10] are exactly all nonisomorphic two-graphs from $E_8 \oplus E_8$.

1 Introduction

A two-graph is a set of *coherent* triples such that every 4-set contains an even number of coherent triples. But it is more important for us that each two-graph is represented by a set of equiangular lines. If the acute angle between lines is equal to $arccos\frac{1}{m}$, then we can choose a vector of norm (squared length) m along each line such that inner products of chosen vectors is equal to ± 1. If v_1, v_2, v_3 are vectors along 3 lines, then these 3 lines compose a coherent triple if $(v_1v_2)(v_2v_3)(v_3v_1) = -1$, where v_iv_j is the inner product of vectors v_i and v_j.

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Sets of equiangular lines of a sufficient large size in a space of fixed dimension exist only if m is an odd number. There is a special bound on the maximal number $n(m,d)$ of equiangular lines at the angle $arccos\frac{1}{m}$ in a space of dimension d:

$$n(m,d) \leq \frac{d(m^2-1)}{m^2-d}.$$

This bound is achieved if and only if the corresponding two-graph is *regular*. A two-graph is regular if every pair of points belongs to the same number of coherent triples.

This is a great problem to classify all regular two-graphs $\mathcal{T}(m,d)$ with parameters m and d, in particular, to find a number $N_m(d)$ of all nonisomorphic two-graphs with the same parameters.

For $m=3$, a regular two-graph $\mathcal{T}(3,d)$ exists only for $d=5,6$ and 7, and $\mathcal{T}(3,d)$ is unique in each dimension, i.e. $N_3(d)=1$ for $d=5,6,7$.

We are interested here in regular two-graphs $\mathcal{T}(5,15)$ with $m=5$ and $d=15$, when $n(5,15)=36$. A set of equiangular lines at angle $arccos\frac{1}{5}$ representing a regular two-graph $\mathcal{T}(5,d)$ may exist only in dimensions $d=5$, 10, 13, 15, 19, 20, 21, 22 and 23. Regular two-graphs $\mathcal{T}(5,d)$ are known for all these $d\neq 19,20$. It is known also the number $N_5(d)$ of all nonisomorphic two-graphs $\mathcal{T}(5,d)$ for $d=5,10,13,23$, namely, $N_5(5)=N_5(10)=N_5(23)=1$, $N_5(13)=4$.

Seidel mentions in [9] at first time results of E.Spence on computing nonisomorphic two-graphs on 36 points. Spence find that $N_5(15)\geq 227$. The number 227 is composed of 11 two-graphs from Latin squares of order 6, of 80 two-graphs from Steiner triple systems of order 15, and 136 new ones discovered by use of a computer [10]. In [9] Seidel describes a large subclass of the Spence' family. Every two-graph of this subclass is related to a 2-(10,4,2) design, and it is shown in [8] that it is a special gluing of the unique two-graphs $\mathcal{T}(5,10)$ and $\mathcal{T}(3,5)$.

We show in this paper that every two-graph $\mathcal{T}(5,15)$ from $E_8\oplus E_8$ is a similar gluing of the unique two-graphs $\mathcal{T}(3,7)$ and $\mathcal{T}(7,7)$. More precisely, a projection of the set of equiangular lines representing a two-graph from $E_8\oplus E_8$ onto a 7-dimensional space X is a set of equiangular lines representing $\mathcal{T}(3,7)$. A restriction of the set of equiangular lines onto another 7-dimensional space orthogonal to X is a set of equiangular lines representing $\mathcal{T}(7,7)$.

An inspection of the list of two-graphs computed by Spence [10] shows that the new two-graphs discovered by Spence which are not related to 2-(10,4,2) designs are two-graphs from $E_8\oplus E_8$.

Denote by a class D and a class E the classes of two-graphs which are gluing $\mathcal{T}(5,10)$ with $\mathcal{T}(3,5)$, and $\mathcal{T}(3,7)$ with $\mathcal{T}(7,7)$, and can be obtained from the lattices D_{16}^+ and $E_8\oplus E_8$, respectively.

All in all we have the following 4 main classes of two-graphs on 36 points:

1) 80 two-graphs from Steiner triple systems on 15 points, the class STS,

2) 11 two-graphs from Latin squares of order 6, the class LSQ,

3) 100 two-graphs of the class D from D_{16}^+,

4) 100 two-graphs of the class E from $E_8\oplus E_8$.