Extremal problems for affine cubes of integers

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Abstract

A collection H of integers is called an *affine d-cube* if there exist $d+1$ positive integers $x_0, x_1, \ldots, x_d$ so that

$$H = \left\{ x_0 + \sum_{i \in I} x_i : I \subseteq \{1, 2, \ldots, d\} \right\}.$$

We address both density and Ramsey-type questions for affine d-cubes.

Regarding density results, upper bounds are found for the size of the largest subset of $\{1, 2, \ldots, n\}$ not containing an affine d-cube. In 1892 Hilbert published the first Ramsey-type result for affine d-cubes by showing that for any positive integers r and d, there exists a least number $n = h(d, r)$ so that for any r-coloring of $\{1, 2, \ldots, n\}$, there is a monochromatic affine d-cube. Improvements for upper and lower bounds on $h(d, r)$ are given for $d > 2$.

1 Introduction

In this section, we give a very brief survey of results related to partition and density problems for affine cubes of integers. For a more detailed survey of these extremal problems, see [12], for example. In Section 2, we state two main results (Theorems 2.1 and 2.2) and contrast them to known bounds. The proofs of these follow in the remaining sections, including some tools using extremal graph theory and a partition theorem for sets of integers not containing any arithmetic progression of length three.

For a set X we use the standard notations $\mathcal{P}(X) = \{Y : Y \subseteq X\}$ and $[X]^s = \{S \subset X : |S| = s\}$. It will often be convenient to use $X = [n] =$

$[1, n] = \{1, 2, \ldots, n\}$. An arithmetic progression of length k will be denoted by AP_k.

1.1 Arithmetic progressions

One of the most well-known (Ramsey-type) partition theorem for sets of integers was published In 1927 [28] van der Waerden published his celebrated partition theorem for integers (see also [29] for history of its proof).

Theorem 1.1 (van der Waerden) *For positive integers d and r, there is a least number $n = W(d, r)$ so that for any partition of $[n]$ into r classes, one class contains an AP_d.*

For more proofs of van der Waerden's theorem we refer the reader to any of, for example, [3], [7], [11], [15], [16], or [24].

At present, there is not much more known about general upper bounds for the function $\mathrm{W}(d, r)$, except that it is primitive recursive (see [24]). Behrend [1] proved a (lower bound) density result for subsets of $[n]$ not containing any AP_3. (See [8] and [9] for related results.)

Theorem 1.2 (Behrend) *There exists a constant c so that for m sufficiently large, there exists an AP_3-free set $B \subset [m]$ with*

$$|B| \geq me^{-c\sqrt{\ln m}} = m^{1-o(1)}.$$

1.2 Affine cubes

In 1892, Hilbert [18] proved the first non-trivial partition Ramsey type theorem. This theorem, which preceded the celebrated Schur and van der Waerden theorems, states that in any finite coloring of the set of positive integers, there exists a monochromatic "affine d-cube" (defined below).

Definition 1.3 *A collection H of integers is called a d-*dimensional affine cube*, or more simply, an* affine d-cube *if there exist $d+1$ positive integers $x_0, x_1, \ldots, x_d$ so that*

$$H = \left\{ x_0 + \sum_{i \in I} x_i : I \subseteq [d] \right\}.$$

If all sums are distinct, that is, if $|H| = 2^d$, then we say that the affine d-cube is replete.