Topology of Dupin hypersurfaces with six distinct principal curvatures *

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Abstract

Let M be a Dupin hypersurface in the unit sphere S^{n+1} with six distinct principal curvatures. The main purpose of the present paper is to show M is either diffeomorphic to $SU(2) \times SU(2)/Q_8$ or homeomorphic to the tube around an embedded 5-dimensional complex Fermat hypersurface $X_5(2)$ in S^{13}, where $Q_8 \subset SU(2) = Sp(1)$ denotes the subgroup $\{\pm 1, \pm i, \pm j, \pm k\}$ and $X_5(2) = \{[z_0, z_1, \cdots z_6] \in CP^6 | z_0^2 + z_1^2 + \cdots + z_6^2 = 0\}$. Moreover, in the former case, all of the focal manifolds are diffeomorphic to $S^3 \times RP^2$; In the latter case, one of the focal manifolds is homeomorphic to $X_5(2)$.

1 Introduction

A closed n-dimensional submanifold in the unit sphere S^{n+1} is called a Dupin hypersurface if the multiplicities of the principal curvatures are constant and if each principal curvature function λ restricted to the leaves of the distribution $T_\lambda = \{x | Ax = \lambda x\}$ is constant [16]. The most important special case is isoparametric hypersurface, i.e., hypersurface with constant principal curvatures. Münzner [11] showed that, for an isoparametric hypersurface in the sphere, the number g of distinct principal curvatures must be 1, 2, 3, 4 or 6. This result was generalized by Thorbergsson [16] to the case of embedded closed Dupin hypersurface.

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In the situation of $g = 1$, 2 or 3, Dupin hypersurfaces are Lie equivalent to isoparametric hypersurfaces. The same conclusion does not hold for $g = 4$ or 6. It is a classical result of E.Cartan [2] that any isoparametric hypersurface with 1, 2 or 3 distinct principal curvatures is a homogeneous space. More precisely, it is either a tube around the veronese surface in S^4 or the tube around the standard embedded projective plane CP^2, HP^2 and QP^2 in S^7, S^{13} and S^{25} respectively. Consequently Dupin hypersurface with 1, 2 or 3 distinct principal curvatures is always diffeomorphic to a homogeneous space.

Using Clifford algebra, a family of infinite nonhomogeneous isoparametric hypersurfaces with four distinct principal curvatures was constructed by Ferus, Karcher and Münzner [6]. They are even not homeomorphic or homotopy equivalent to homogeneous spaces [18]. The topology and various restriction about the multiplicities of the principal curvatures of isoparametric hypersurfaces were explored in [1][4].

For the last case, i.e., $g = 6$, the multiplicities of the six distinct principal curvature are the same(we denote it by m) and equal 1 or 2 [7][1]. Dorfmeister and Naher [3] showed that isoparametric hypersurface with $g = 6$ and $m = 1$ is the homogeneous space $SU(3)/T^2$ by solving the Cartan-Münzner equations. Their methods can not work in the case of $m = 2$ or for Dupin hypersurface. Thus it is very natural to ask whether there exists nonhomogeneous Dupin hypersurface with $g = 6$ in geometry or topology.

The present paper is going to handle this question in topology. We will show that each Dupin hypersurface with $g = 6$ is either diffeomorphic to the homogeneous space $SU(2) \times SU(2)/Q_8$ or homeomorphic to the normal sphere bundle of an embedded $X_5(2)$ in S^{13} corresponding to $m = 1$ or 2. Moreover, we prove in the appendix that there are at most 3 different Dupin hypersurfaces with $g = 6, m = 2$ up to homeomorphism. We remind the, where $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$ is the subgroup of the quaternion $SU(2) = Sp(1)$. Except the homogeneous example G_2/T^2, we have no other example of Dupin hypersurface with $g = 6, m = 2$ and with different homeomorphism type. To be honest, I don't know how to prove that each Dupin hypersurface with $g = 6, m = 2$ is homeomorphic to G_2/T^2.

For reader's convenience, we recall the definition and the main properties of complex projective hypersurfaces as the follows.

Let $X_n(2) \subset \mathrm{CP}^{n+1}$ be the Fermat hypersurface of degree 2, i.e, $X_n(2) = \{(z_0, z_1, \cdots z_{n+1}) \in \mathrm{CP}^{n+1} | z_0^2 + z_1^2 + \cdots z_{n+1}^2 = 0\}$. The Lefschtz hyperplane section theorem and Poincaré duality implies that

$$H^i(X_n(2)) \cong Z \text{ if } i \text{ even };\ H^i(X_n(2)) \cong Z \text{ if } i \text{ odd };$$

Moreover, if n is odd, one obtain the cohomology ring $H^*(X_n(2)) \cong Z[x, z]/\{x^j = 2z\}$ here $n = 2j - 1$ and $\deg x = 2$, $\deg z = 2j$. The first Pontryagin class of $X_n(2)$ equal $(n-2)x^2$.