Splitting Patterns and Trace Forms

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Abstract. The splitting pattern of a quadratic form q over a field k consists of all distinct Witt indices that occur for q over extension fields of k. In small dimensions, the complete list of splitting patterns of quadratic forms is known. We show that *all* splitting patterns of quadratic forms of dimension at most nine can be realized by trace forms.

1. Constraints on splitting patterns

Let q be an anisotropic quadratic form of dimension $n \geq 2$ over a field k of characteristic $\neq 2$. Let

$$(i_0(q) = 0, i_1(q), i_2(q), \ldots, i_{h(q)}(q) = [n/2])$$

denote the *splitting pattern* of q; that is, $i_0(q) < i_1(q) < \cdots < i_{h(q)}(q)$ are all distinct Witt indices that occur for q over arbitrary field extensions of k. We refer to M. Knebusch's originating work on generic splitting of quadratic forms, [12], [13]. The index $i_j(q)$ occurs over the field K_j, $0 \leq j \leq h(q)$, in a *generic splitting tower* $K_0 = k, K_1 = k(q), \ldots, K_{h(q)}$ of q over k.

In general, it is still a wide open problem to determine all tuples of strictly increasing integers that are splitting patterns of quadratic forms. However, for an *excellent* quadratic form q over k, the splitting pattern is known in arbitrary dimension, [12, Section 7], [8, Cor. 2.8]. It depends only on the dimension n of q; we denote it by

$$(i_0(n) = 0, i_1(n), \ldots, i_{h(n)}(n) = [n/2])$$

and have for the smallest positive Witt index $i_1(n)$:

1.1 Example. Let q be an anisotropic excellent quadratic form of dimension n over k. Write $n = 2^r + d$, where 2^r is the largest power of 2 less than n. Then the Witt index $i_1(n)$ of q over its function field $k(q)$ is given by

$$i_1(n) = d.$$

For arbitrary anisotropic forms we recall from [6], [9, Cor. 1.13]:

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1.2 Theorem. *Let q be an anisotropic quadratic form of dimension n over k. Then:*

$$i_1(q) \leq i_1(n).$$

In particular, we have by 1.1 and 1.2 for any field k:

1.3 Corollary. *Let q be an anisotropic quadratic form of dimension $n = 2^r + 1$ over k. Then:*

$$i_1(q) = 1.$$

It is natural to ask whether the inequality in 1.2 generalizes for higher Witt indices. Consider anisotropic quadratic forms q of dimension n with $h(n) \geq 2$. In other words, $n \neq 2^r, 2^r - 1$, and this implies $h(q) \geq 2$ by the characterization of all forms q of height $h(q) = 1$ in [12, Thm. 5.8], [18]. So, the indices $i_2(n)$ and $i_2(q)$ are defined, and we obtain:

1.4 Proposition. *Let q be an anisotropic quadratic form over k of dimension n with $h(n) \geq 2$. If $i_1(q) = i_1(n)$, then*

$$i_2(q) \leq i_2(n).$$

Proof. Let q_1 denote the anisotropic kernel of q over its function field $k(q)$. We have

$$i_2(q) = i_1(q) + i_1(q_1).$$

On the other hand,

$$i_2(n) = i_1(n) + i_1(n - 2i_1(n))$$

since $n - 2i_1(n)$ is the dimension of the anisotropic kernel of an n-dimensional anisotropic excellent quadratic form over its function field.

By assumption, $i_1(q) = i_1(n)$. Thus our claim $i_2(q) \leq i_2(n)$ amounts to showing $i_1(q_1) \leq i_1(n - 2i_1(n))$. Again by the assumption, the last inequality is equivalent to

$$i_1(q_1) \leq i_1(n - 2i_1(q));$$

that is,

$$i_1(q_1) \leq i_1(\dim q_1),$$

which is the result of Thm. 2.1 applied to the anisotropic from q_1 over $k(q)$. $\square$

By [9, Cor. 1.13], our assumption $i_1(q) = i_1(n)$ in Prop. 1.4 implies that q becomes an anisotropic Pfister neighbor over some field extension of k. The proof of the proposition carries over to the analogous result for higher Witt indices:

1.5 Corollary. *Let q be an anisotropic quadratic form over k of dimension n with min $\{h(n), h(q)\} \geq j + 1$. If $i_j(q) = i_j(n)$, then*

$$i_{j+1}(q) \leq i_{j+1}(n).$$