Twisted Pfister Forms

DETLEV W. HOFFMANN*
Current Address: Aindorferstr. 84, D-80689 München, Germany

Abstract. Let F be a field of characteristic $\neq 2$. In this paper we investigate quadratic forms φ over F which are anisotropic and of dimension 2^n, $n \geq 2$, such that in the Witt ring WF they can be written in the form $\varphi = \sigma - \pi$ where σ and π are anisotropic n- resp. m-fold Pfister forms, $1 \leq m < n$. We call these forms twisted Pfister forms. Forms of this type with $m = n - 1$ are of great importance in the study of so-called good forms of height 2, and such forms with $m = 1$ also appear in Izhboldin's recent proof of the existence of n-fold Pfister forms τ over suitable fields F, $n \geq 3$, for which the function field $F(\tau)$ is not excellent. We first derive some elementary properties and try to give alternative characterizations of twisted Pfister forms. We also compute the Witt kernel $W(F(\varphi)/F)$ of a twisted Pfister form φ. Our main focus, however, will be the study of the following problems: For which forms ψ does a twisted Pfister form φ become isotropic over $F(\psi)$? Which forms ψ are equivalent to φ (i.e., the function fields $F(\varphi)$ and $F(\psi)$ are place-equivalent over F)? We also investigate how such twisted Pfister forms behave over the function field of a Pfister form of the same dimension which then leads to a generalization of Izhboldin's results.

1. Introduction

Let F be a field of characteristic $\neq 2$. WF denotes the Witt ring of non-degenerate quadratic forms over F (which we will simply call forms over F). P_nF (resp. GP_nF) denotes the set of all forms isometric (resp. similar) to n-fold Pfister forms, i.e., forms of the type $\langle\langle a_1, \cdots, a_n\rangle\rangle = \langle 1, a_1\rangle \otimes \cdots \langle 1, a_n\rangle$. We say that φ is a Pfister neighbor if there exists $\pi \in P_nF$ for some n such that φ is similar to a subform of π and $\dim\varphi > \frac{1}{2}\dim\pi = 2^{n-1}$. In this case, we say that φ is a Pfister neighbor of π. Any $\pi \in P_nF$ can be written as $\pi \simeq \langle 1\rangle \perp \pi'$. π' is called the pure part of π and it is uniquely determined up to isometry.

An important part of the algebraic theory of quadratic forms deals with the behaviour of forms over F under a field extension K/F. Of particular interest is the case where $K = F(\psi)$ is the function field of a form ψ over F. If ψ is isotropic then $F(\psi)/F$ is purely transcendental. This situation is of not much interest with regard to the questions we will consider since one of our main goals lies in determining whether an anisotropic form φ over F becomes isotropic over K, something which cannot happen if

*This research has been carried out during stays at the Mathematics Department of the University of Kentucky, Lexington and the Sonderforschungsbereich 343: "Diskrete Strukturen in der Mathematik", Universität Bielefeld. I would like to thank both institutions for their hospitality.

K/F is purely transcendental. The extension K/F is said to be excellent if for any form φ over F the anisotropic part $(\varphi_K)_{\text{an}}$ of φ over K is defined over F, i.e., there exists a form $\tilde{\varphi}$ over F such that $(\varphi_K)_{\text{an}} \simeq \tilde{\varphi}_K$. Knebusch has shown in [K 2, Theorem 7.13] that if $F(\psi)/F$ is excellent where ψ is an anisotropic form, then ψ is a Pfister neighbor. As for the converse of this statement, it suffices to consider Pfister forms. This is because if ψ is a Pfister neighbor of τ then $F(\psi)$ and $F(\tau)$ are (place-)equivalent over F which implies that $F(\psi)$ is excellent iff $F(\tau)$ is excellent. So let $K = F(\tau)$ for some anisotropic $\tau \in P_nF$. It is easy to show that K/F is excellent for $n = 1$, and for $n = 2$ this was shown by Arason in [ELW 1, Appendix II]. It was an open problem whether K/F is always excellent for $n \geq 3$ until recently, when Izhboldin [I] showed that in this case there exists always a field extension E/F, some $\sigma \simeq \langle 1\rangle \perp \sigma' \in P_nE$ and some $d \in \dot{E} = E\setminus\{0\}$ not a square such that $\sigma' \perp \langle d\rangle$ is anisotropic, it becomes isotropic over $E(\tau)$, but its anisotropic part over $E(\tau)$ is not defined over E. In particular, $E(\tau)/E$ is not excellent.

Let us now turn to a seemingly unrelated problem. It is well-known that if φ is an anisotropic form over F then $\varphi_{F(\varphi)}$ is hyperbolic iff $\varphi \in GP_nF$ for some n. Going one step further, what can one say about an anisotropic form φ over F for which $\varphi_1 \simeq (\varphi_{F(\varphi)})_{\text{an}}$ does not vanish but where $(\varphi_1)_{F_1(\varphi_1)}$ becomes hyperbolic where $F_1 = F(\varphi)$. Such a form is said to be of height 2. By the above, we know that $\varphi_1 \in GP_mF_1$ for some $m \geq 1$ and we say that φ has degree m. We call φ good if there exists some $\rho \in P_mF$ such that $\varphi_1 \simeq a\rho_{F_1}$ for some $a \in \dot{F}_1$. If one can choose $a \in \dot{F}$ already then φ is an excellent form in the sense of Knebusch [K 2, Section 7], and in this case one knows how φ has to look like (cf. [K 2, Lemma 10.1(i)]). An open problem is to classify anisotropic good non-excellent forms of height 2. It is believed that if φ is of that type and of degree $n-1$ then there exists some $\alpha \in P_{n-2}F$ and some 4-dimensional form β over F such that $\varphi \simeq \alpha \otimes \beta$ and $\alpha \otimes \langle\langle -d\rangle\rangle$ is anisotropic where $d = d_{\pm}\beta$ is the signed discriminant of β. This conjecture has been proved for $n = 2$ (cf. [K 2, Theorem 10.3]), $n = 3$ (cf. [F 2, Theorem 1.6]), and $n = 4$ (cf. [Ka, Théorème 2.12]). (It is easy to show that if φ is of this type $\alpha \otimes \beta$ then φ is good non-excellent of height 2.)

What do these forms $\alpha \otimes \beta$ of height 2 and Izhboldin's examples $\sigma' \perp \langle d\rangle$ have in common? In both cases we are dealing with anisotropic forms φ of dimension 2^n. If $\varphi \simeq \alpha \otimes \beta$ and if we write $\beta \simeq \langle d, u, v, uv\rangle$ (possibly after scaling), then in WF we have

$$\varphi = \alpha \otimes \langle\langle u, v\rangle\rangle - \alpha \otimes \langle\langle -d\rangle\rangle \ .$$

If $\varphi \simeq \sigma' \perp \langle d\rangle$ then in WF we have

$$\varphi = \sigma - \langle\langle -d\rangle\rangle \ .$$

We observe that in these situations φ can be written as the difference of an n-fold Pfister form and an $(n-1)$-fold resp. 1-fold Pfister form. We aim at a unifying concept which includes both types of forms $\alpha \otimes \beta$ and $\sigma' \perp \langle d\rangle$. This leads us quite naturally to what we call *twisted Pfister forms*. φ is said to be a twisted Pfister form if φ is anisotropic of dimension 2^n for some n, such that in WF it can be written as $\varphi = \sigma - \pi$ for some anisotropic forms $\sigma \in P_nF$ and $\pi \in P_mF$, $1 \leq m < n$. The above examples represent twisted Pfister forms at the extreme ends of the spectrum: $m = n-1$ and $m = 1$. These examples also serve as a motivation for our in-depth study of these forms.

As simple as the structure of twisted Pfister forms appears, this class of forms leads in our opinion to a wealth of interesting results and new problems as the above examples