Nonparametric Wavelet based Minimax Estimation in non Gaussian Models

F. GÖTZE AND B.A. ZALESKY[1]

Abstract

We investigate nonparametric estimation procedures for images and functions with discontinuities in the presence of additive noise using wavelet decomposition methods. For functions and images belonging to Besov type spaces we derive bounds for the minimax risk in the class of linear and the class of arbitrary estimators for *non Gaussian* additive error models. The bounds are sharp for the linear minimax risks under second order moment assumptions on the noise. They are sharp up to a logarithmic factor for the general minimax risks under exponential moment conditions for the noise. Applications are made to quantization and image compression problems.

Key Words and Phrases. Haar bases, wavelets, minimax risk, linear estimators, threshold estimators, quantization errors.

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1 Introduction

In this paper the properties of the linear estimators and non-linear wavelet threshold estimators of functions (or images) are studied in the presence of *non Gaussian* additive noise. Bounds for the minimax risks are derived which are based on the tail and local dependence structure of the error models.

Let $I = [0,1]^d$ denote the unit cube of the Euclidean space R^d. For an integer r and a displacement $h \in R^d$ let $\Delta_h^r(f(x))$ denote the r-th order difference of f taken at the points $x + lh, l = 0, \ldots, r$. Using these differences one can define the r-th modulus of smoothness in $L^p(I)$ as

$$w_r(f,t)_p \stackrel{\text{def}}{=} \left(\sup_{|h| \le t} \int_I |\Delta_h^r(f(x))|^p \mathbf{1}_{\{x+rh \in I\}} dx \right)^{1/p},$$

and the Besov space $B_{p,q}^{\alpha,r}, r < \alpha + 1$ as the set of equivalence classes of measurable functions such that

$$||f||_{B_{p,q}^{\alpha,r}} \stackrel{\text{def}}{=} ||f||_{L^p(I)} + \left(\int_0^\infty \left[t^{-\alpha} w_r(f,t)_p \right]^q t^{-1} dt \right)^{1/q}$$

is finite.

Let $J_m \in I$ be dyadic cubes with sidelength 2^{-m}. Denote by D_m the set of all J_m and for some previously fixed integer l the union $D = \bigcup_{m=0}^l D_m$. We shall consider functions f belonging to subspaces N_l resp. $\bar{N}_l$ of the Besov spaces $B_{p,q}^{\alpha,r}$ resp. $B_{\tau,\tau}^{\alpha,r}$, $\tau = (\alpha/d + 1/2)^{-1}$, generated by the elements up to resolution 2^{-l} of an orthogonal compactly supported wavelet basis $\phi_{J_m}(x) = \phi(2^m x - j)$ i.e.

$$f = \sum_{m=0}^{l} \sum\nolimits_{J_m \in D_m} \beta_{J_m} \phi_{J_m}.$$

We shall investigate functions sampled with random noise ξ, i.e. we study the discrete time random processes y indexed by J_l given by the coefficients β_{J_l} sampled with random noise ξ_{J_l}

$$y_{J_l} = \beta_{J_l} + \xi_{J_l}.$$

Let ξ_{J_l} denote mean zero random variables with finite second moments $\mathbf{E}\xi_{J_l}^2 < \infty$ which are not necessary identically distributed. Let $B_{l,M}$ resp. $\bar{B}_{l,M}$ denote the Besov balls in N_l resp. $\bar{N}_l$ defined by $||f||_{B_{p,q}^{\alpha,r}} < M$ resp. $||f||_{B_{\tau,\tau}^{\alpha,r}} < M$. We shall investigate the minimax risks

$$\mathbf{R}_L(M,\Theta) = \sup_{\xi \in \Theta} \inf_{\hat{f}-\text{linear}} \sup_{B_{l,M}} \mathbf{E}||f - \hat{f}||_{L^2(I)}^2,$$

in the class of linear estimators, and

$$\mathbf{R}_{NL}(M,\Theta) = \sup_{\xi \in \Theta} \inf_{\hat{f}} \sup_{\bar{B}_{l,M}} \mathbf{E}||f - \hat{f}||_{L^2(I)}^2,$$