Generic modules over artin algebras

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Generic modules have been introduced by Crawley-Boevey in order to provide a better understanding of finite dimensional algebras of tame representation type. In fact he has shown that the generic modules correspond to the one-parameter families of indecomposable finite dimensional modules over a tame algebra [5]. The Second Brauer-Thrall Conjecture provides another reason to study generic modules because the existence of a generic module over an artin R-algebra Λ ($R/\operatorname{rad} R$ an infinite field) implies that Λ has strongly unbounded representation type, i.e. there are infinitely many pairwise non-isomorphic Λ-modules of length n for infinitely many $n \in \mathbb{N}$ [6].

The aim of this paper is to develop further the analysis of existence and properties of generic modules. Our approach depends to a large extent on the embedding of a module category into a bigger functor category. These general concepts are explained in the first two sections. We continue in Section 3 with a new characterization of the pure-injective modules which occur as the source of a minimal left almost split morphism. This is of interest in our context because generic modules are pure-injective. Next we consider indecomposable endofinite modules. Recall that a module is endofinite if it is of finite length when regarded in the natural way as a module over its endomorphism ring. Changing slightly the original definition, we say that a module is generic if it is indecomposable endofinite but not finitely presented. Section 4 is devoted to several characterizations of generic modules in order to justify the choice of the non-finitely presented modules as the generic objects. We prove them for dualizing rings, i.e. a class of rings which includes noetherian algebras and artinian PI-rings. Existence results for generic modules over dualizing rings follow in Section 5. Several results in this paper depend on the fact that a functor $f\colon \operatorname{Mod}(\Gamma) \to \operatorname{Mod}(\Lambda)$ which commutes with direct limits and products, preserves certain finiteness conditions. For example, if a Γ-module M is endofinite then $f(M)$ is endofinite. If in addition $\operatorname{End}_\Gamma(M)$ is a PI-ring, then $\operatorname{End}_\Lambda(N)$ is a PI-ring for every indecomposable direct summand N of $f(M)$. This material is collected in Section 6 and 7. In Section 8 we introduce an effective method to construct generic modules over artin algebras from so-called generalized tubes. The special case of a tube in the Auslander-Reiten quiver is discussed in the following section. We obtain as a consequence a parametrization of the stable tubes by indecomposable Σ-pure-injective modules. These results are combined with Crawley-Boevey's work on tame algebras. It is shown that for a tame algebra Λ and a fixed $n \in \mathbb{N}$ almost all n-dimensional indecomposable Λ-modules belong to a disjoint union $\cup_G \operatorname{ind}_G(\Lambda)$ of naturally defined sets such that each $\operatorname{ind}_G(\Lambda)$ corresponds to a generic module G. Moreover we present a new characterization of tameness in terms of homogeneous tubes in the Auslander-Reiten quiver. We continue in Section 10 with a discussion of representation embeddings. A pure representation embedding $\operatorname{Mod}(\Gamma) \to \operatorname{Mod}(\Lambda)$ induces a well-behaved map between the corresponding Ziegler spectra and we apply this to algebras of wild representation type. The final section is devoted to the Krull dimension of the category of finitely presented modules over an artin algebra. We obtain a characterization of those algebras having dimension 1 which allows one to deduce that this cannot occur for a finite dimensional algebra over an algebraically closed field.

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1. Filtrations and dimensions for abelian categories

Let $\mathcal{C}$ be an abelian category. Recall that a full subcategory $\mathcal{S}$ of $\mathcal{C}$ is a *Serre subcategory* provided that for every exact sequence $0 \to X' \to X \to X'' \to 0$ in $\mathcal{C}$ the object X belongs to $\mathcal{S}$ iff X' and X'' belong to $\mathcal{S}$. For every Serre subcategory $\mathcal{S}$ of $\mathcal{C}$ one can form the *quotient category* $\mathcal{C}/\mathcal{S}$ as follows. The objects of $\mathcal{C}/\mathcal{S}$ are those of $\mathcal{C}$ and $\operatorname{Hom}_{\mathcal{C}/\mathcal{S}}(X,Y) = \varinjlim \operatorname{Hom}_{\mathcal{C}}(X', Y/Y')$ with $X' \subseteq X$, $Y' \subseteq Y$ and $X/X', Y' \in \mathcal{S}$. Again the category $\mathcal{C}/\mathcal{S}$ is abelian and there is canonically defined the *quotient functor* $q\colon \mathcal{C} \to \mathcal{C}/\mathcal{S}$ such that $q(X) = X$; it is exact with $\operatorname{Ker}(q) = \mathcal{S}$ [7]. Here the *kernel* $\operatorname{Ker}(f)$ of a functor $f\colon \mathcal{C} \to \mathcal{D}$ is, by definition, the full subcategory of all objects X such that $f(X) = 0$. If f is exact, then $\operatorname{Ker}(f)$ is automatically a Serre subcategory and there is induced an exact faithful functor $\mathcal{C}/\operatorname{Ker}(f) \to \mathcal{D}$. A Serre subcategory is called *localizing* provided that the corresponding quotient functor admits a right adjoint. For each object X in $\mathcal{C}$ we denote by $\mathsf{L}(X)$ the lattice of its subobjects in $\mathcal{C}$; it is a modular lattice with 0 and 1.

Now suppose that $\mathcal{L}$ is a class of modular lattices which is closed under intervals, quotients