Block P-Matrices

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Abstract.
Several possibilities to generalize P-matrices to block form are discussed. Unfortunately certain equivalences holding for P-matrices do not carry over to the block case. We opt for one such generalization calling it block P-matrices . It has the most analogies to the usual P-matrices .

This class and its connections to other generalizations is studied in some detail.

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1 Introduction

From the point of view of applicability, real P-matrices are very interesting. We refer to their role in the linear complementarity problem (see [1]), in global invertibility of vector fields [2] and in nonsingularity characterization of interval matrices [6]. We recall that a square matrix is called a P-*matrix* if all its principal minors are positive.

In [4] Johnson and Tsatsomeros gave a new characterization of this class: an n-by-n real matrix A is a P-matrix if and only if the set $r(A)$ consisting of all matrices of the form $TA + (I - T)$, where $T = \operatorname{diag}(t_1, \ldots, t_n)$ with $t_i \in [0, 1]$, is nonsingular, i.e., if each matrix in $r(A)$ is nonsingular. Besides this they also give a characterization of all matrices A such that $tA+(1-t)I$ is nonsingular for all $t \in [0, 1]$. This suggests to study this problem for matrices T "between" diagonal matrices and scalar matrices (multiples of the identity matrix), namely those which are direct products of scalar matrices for a given partition of $\{1, \ldots, n\}$. Consider the following example.

Example. Let

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 2 & 1 \end{bmatrix}$$

and let $T = \operatorname{diag}(t_1, t_2, t_2)$ with $t_1, t_2 \in [0, 1]$. Then a direct calculation yields

$$\det(TA + (I - T)) = 1 - t_2 + 2t_2^2 + t_1 t_2$$

which is always positive in the given range.

As $a_{22} = 0$, A is not a P-matrix but $TA + (I - T)$ is nonsingular for each T from the "restricted" set of T's.

This example motivates us to introduce a new matrix class below and investigate how it relates to the class of P-matrices .

2 Notation and preliminaries

Let $N(k)$ be a partition of the set $N = \{1, \ldots, n\}$ into k, $1 \le k \le n$, pair-wise disjoint subsets N_j of the cardinality n_j and let

$$\mathcal{M}_k = \{\bigcup_{j\in\alpha} N_j : \ \alpha \subseteq \{1, \ldots, k\}\}.$$