Absolutely continuous flows generated by Sobolev class vector fields in finite and infinite dimensions

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Abstract

We prove the existence of the global flow $\{U_t\}$ generated by a vector field A from a Sobolev class $W^{1,1}(\mu)$ on a finite or infinite dimensional space X with a measure μ, provided μ is sufficiently smooth, and ∇A and $|\delta_\mu A|$ (where $\delta_\mu A$ is the divergence with respect to μ) are exponentially integrable. In addition, the measure μ is shown to be quasi-invariant under $\{U_t\}$. In the case $X = \mathbb{R}^n$ and $\mu = pdx$, where $p \in W^{1,1}_{\text{loc}}(\mathbb{R}^n)$ is a locally uniformly positive probability density, a sufficient condition is: $\exp(c\|\nabla A\|) + \exp(c|(A, \frac{\nabla p}{p})|) \in L^1(\mu)$ for all c. In the infinite dimensional case we get analogous results for measures differentiable along sufficiently many directions. Examples of measures which fit our framework, important for applications, are symmetric invariant measures of infinite dimensional diffusions and Gibbs measures. Typically, in both cases such measures are essentially non-Gaussian. Our result in infinite dimensions significantly extends previously studied cases where μ was a Gaussian measure. Finally, we study flows generated by vector fields whose values are not necessarily in the Cameron–Martin space.

Keywords: Flows; Absolute continuity; Divergence; Logarithmic derivative; Sobolev classes
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Contents

1 Introduction

One of the fundamental problems of the theory of dynamical systems concerns the existence and uniqueness of a global flow $\{U_t\}$ generated by a vector field A. It is often desirable to have a measure μ which is invariant or quasi-invariant under $\{U_t\}$. A standard result says that if A is uniformly Lipschitzian on $\mathbb{R}^n$ or on a Banach space, then there is a unique flow $\{U_t\}$ generated by A. In addition, in the finite dimensional case the Lebesgue measure is quasi-invariant under $\{U_t\}$ (hence this is true for every measure with strictly positive density). However, in many applications, e.g., in fluid mechanics, statistical physics, stochastic analysis (especially in infinite dimensions) one has to deal with vector fields which are not even locally Lipschitzian, but still have certain regularity properties such as belonging to some Sobolev class. This kind of regularity not always implies continuity, so the very existence of solutions is not granted in advance. Recent infinite dimensional analysis research, in particular in infinite dimensional manifolds, gives rise to questions of the same sort (see, e.g., [21], [22], [26], [28], [31]).

Even if we deal with uniformly bounded continuous fields on $\mathbb{R}^n$ it may happen that such a weak regularity is not enough, e.g., for uniqueness. An obvious example: $n = 1$, $A(x) = 0$ on $(-\infty, 0]$, $A(x) = \min(x^\alpha, 1)$ on $[0, \infty)$, with $\alpha \in (0, 1)$. Then A is in $W^{p,1}_{\text{loc}}(\mathbb{R}^1)$ for $p < \frac{1}{1-\alpha}$, but the corresponding equation has many solutions. It is shown in [25] that for each $p > 1$ there is a field $A \in C_b(\mathbb{R}^2) \cap W^{p,1}_{\text{loc}}(\mathbb{R}^2)$ generating infinitely many flows (that is, solutions with the group property) under which the Lebesgue measure is quasi-invariant. It also worth mentioning that in infinite dimensions the continuity of A does not imply even local solvability (see, e.g., [8, Section 2]).

Before proceeding further let us state with some precision what we mean by a solution to our dynamical systems: