CONVERGENCE OF OPERATOR SEMIGROUPS GENERATED BY ELLIPTIC OPERATORS

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Abstract. Consider a sequence of operator semigroups $\left(T_t^{(n)}\right)_{t>0}$, $n \in \mathbb{N}$, whose generators are elliptic and are (informally) of type $L^{(n)} = \sum_{i,j=1}^d \partial_i \left(a_{ij}^{(n)} \partial_j + d_i^{(n)}\right) - \sum_{i=1}^d b_i^{(n)} \partial_i - c^{(n)}$ on a possibly unbounded open set $U \subset \mathbb{R}^d$ with measurable coefficients. Under weak assumptions on the coefficients we prove strong convergence of $\left(T_t^{(n)}\right)_{t>0}$, $n \in \mathbb{N}$, on $L^2(U; dx)$ resp. the Sobolev space $H_0^{1,2}(U; dx)$. In particular, this is done without assuming that $b_i^{(n)}, d_i^{(n)}, c^{(n)}$, are bounded in $L^\infty(U; dx)$ uniformly in n.

1. Introduction and main results

Let $U \subset \mathbb{R}^d$, $d \geq 3$, U open (not necessarily bounded), and let dx denote Lebesgue measure on U. Below all functions are supposed to be real–valued. Let $a_{ij}^{(n)}$, $b_i^{(n)}$, $d_i^{(n)}$, $c^{(n)} \in L^1_{\text{loc}}(U; dx)$, $1 \leq i, j \leq d$, $n \in \mathbb{N} \cup \{\infty\}$ satisfying the following conditions:

(1.1) There exists $\delta \in]0, \infty[$ such that for all $n \in \mathbb{N} \cup \{\infty\}$ and dx–a.e. $x \in U$

$$\sum_{i,j=1}^d a_{ij}^{(n)}(x) \xi_i \xi_j \geq \delta \sum_{i=1}^d \xi_i^2 \text{ for all } \xi_1, \ldots, \xi_d \in \mathbb{R} .$$

(1.2) There exists $M \in [0, \infty[$ such that for all $n \in \mathbb{N}$ and dx–a.e. $x \in U$

$$\left| a_{ij}^{(n)}(x) \right| \leq M \ , \ 1 \leq i, j \leq d .$$

(1.3) There exist $p_{b,i}$, $p_{d,i}$, $p_c \in [d, \infty]$, $1 \leq i \leq d$, such that for all $n \in \mathbb{N} \cup \{\infty\}$

$$b_i^{(n)} \in L^{p_{b,i}}(U; dx) \ , \ d_i^{(n)} \in L^{p_{d,i}}(U; dx) \ , \ c^{(n)} \in L^{p_c/2}(U; dx) .$$

Note that (1.1) is a condition only on the symmetric part of $(a_{ij})_{1 \leq i,j \leq d}$. Conditions (1.1) – (1.3) allow to construct the corresponding coercive closed forms (cf. e.g. [MR

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92, Chap. I, Sect. 2]) as follows. Let $C_0^\infty(U)$ denote the set of all infinitely differentiable functions with compact support in U. Fix $n \in \mathbb{N} \cup \{\infty\}$ and set $\partial_i := \frac{\partial}{\partial x_i}$, $1 \le i \le d$. Define

(1.4)

$$\mathcal{E}^{(n)}(u,v) := \sum_{i,j=1}^{d} \int \partial_i u \ \partial_j v \ a_{ij}^{(n)} \, dx + \sum_{i=1}^{d} \int u \ \partial_i v \ d_i^{(n)} \, dx$$
$$+ \sum_{i=1}^{d} \int \partial_i u \ v \ b_i^{(n)} \, dx + \int u \ v \ c^{(n)} \, dx \ ; \ u,v \in C_0^\infty(U) \ .$$

For $\alpha \in]0,\infty[$ set

$$\mathcal{E}_\alpha^{(n)}(u,v) := \mathcal{E}^{(n)}(u,v) + \alpha(u,v)_{L^2(U;dx)}; u,v \in C_0^\infty(U) \ .$$

E.g. by [MR 93, Theorem 2.2] we know that there exists $\alpha_n \in]0,\infty[$ such that $\left(\mathcal{E}_{\alpha_n}^{(n)}, C_0^\infty(U)\right)$ is closable on $L^2(U;dx)$ and its closure $\left(\mathcal{E}_{\alpha_n}^{(n)}, D(\mathcal{E}_{\alpha_n}^{(n)})\right)$ is a coercive closed form on $L^2(U;dx)$ in the sence of [MR 92, Chap. I, Definition 2.4]. It is well–known (and can e.g. easily be extracted from the proof of [MR 93, Theorem 2.2], or more precisely from the proof of the underlying [RSch 95, Theorem 1.7]) that there exist $\gamma_n \in]0,\infty[$ such that for all $u,v \in C_0^\infty(U)$

$$\left|\mathcal{E}_{\alpha_n}^{(n)}(u,v)\right| \le \gamma_n \ \mathcal{E}_{\alpha_n}^{(n)}(u,u)^{1/2} \mathcal{E}_{\alpha_n}^{(n)}(v,v)^{1/2} \tag{1.5}$$

$$\gamma_n^{-1} \ |u|_{1,2} \ \le \mathcal{E}_{\alpha_n}^{(n)}(u,u)^{1/2} \le \gamma_n \ |u|_{1,2} \ . \tag{1.6}$$

Here $|\ |_{1,2}$ is the norm on the classical Sobolev space $H_0^{1,2}(U;dx)$ of order 1 in $L^2(U;dx)$, defined as the completion of $C_0^\infty(U)$ w.r.t. $|\ |_{1,2}$ which is given by

$$|u|_{1,2}^2 := \sum_{i=1}^{d} \int (\partial_i u)^2 \, dx + \int u^2 \, dx \ ; \ u \in C_0^\infty(U) \ .$$

In particular, $D\left(\mathcal{E}_{\alpha_n}^{(n)}\right) = H_0^{1,2}(U,dx)$ and (1.5), (1.6) hold for all $u \in H_0^{1,2}(U;dx)$.

Remark 1.1. γ_n in (1.5), (1.6) only depends on α_n, δ, M and the L^p–norms of $b_i^{(n)}$, $d_i^{(n)}$, $c^{(n)}$, $1 \le i \le d$, (cf. condition (1.3)). This can also be seen e.g. from the respective proofs in [MR 93], [RSch 95] mentioned above. In particular, α_n and γ_n can be chosen to be independent of n, if all the L^p–norms in condition (1.3) are bounded uniformly in n.

Let $(L_{\alpha_n}, D(L_{\alpha_n}))$, $\left(T_{\alpha_n,t}\right)_{t>0}$ be the generator resp. the strongly continuous contraction semigroup associated with $\left(\mathcal{E}_{\alpha_n}^{(n)}, D(\mathcal{E}_{\alpha_n}^{(n)})\right)$ (cf. e.g. [MR 92, Chap. I., Sect. 2]). Define

$$T_t^{(n)} := e^{\alpha_n t} T_{\alpha_n,t} \ , \ t > 0 \ , \tag{1.7}$$

$$L^{(n)} := L_{\alpha_n} + \alpha_n \ , \ D(L^{(n)}) := D(L_{\alpha_n}) \ . \tag{1.8}$$

Then $(L^{(n)}, D(L^{(n)})$ generates $(T_t^{(n)})_{t>0}$ (on $L^2(U;dx)$.