Intersection statements for systems of sets

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Abstract

A family of r sets is called a Δ-*system* if any two sets have the same intersection. Denote by $F(n,r)$ the most number of subsets of an n-element set which do not contain a Δ-system consisting of r sets. Constructive new lower bounds for $F(n,r)$ are given which improve known probabilistic results, and a new upper bound is given by employing an argument due to Erdős and Szemerédi. Another construction is given which shows that for certain n, $F(n,3) \geq 1.551^{n-2}$. We also show a relationship between the upper bound for $F(n,3)$ and the Erdős–Rado conjecture on the largest uniform family of sets not containing a Δ-system.

1 Introduction

A family $\mathcal{F}$ of sets is called k-*uniform* if for every $F \in \mathcal{F}$, $|F| = k$ holds. A family of sets is called a Δ-*system* if any two sets have the same intersection. Define $f(k,r)$ to be the least integer so that any k-uniform family of $f(k,r)$ sets contains a Δ-system consisting of r sets. Erdős and Rado [8] proved that

$$(r-1)^k < f(k,r) < k!(r-1)^k \tag{1}$$

and conjectured that for each r, there exists a constant C_r so that $f(k,r) < C_r^k$. Erdős (see [6]) has offered 1000 dollars for the proof or disproof of this

*This work was partially supported by the Network DIMANET/PECO of the European Union (ERBCIPDCT 940623) and the grant RPY300 of International Science Foundation and Russian Government.

for $r = 3$. Several authors (Abbott, Hanson, and Sauer [3], Abbott and Hanson [4], Spencer [14], and Kostochka [12, 13]) have slightly improved the bounds in (1) but a proof or disproof of the conjecture is nowhere in sight. Currently, the best known upper bound [13] is

$$f(k,r) < Ck!\left(\frac{(\log\log\log k)^2}{\alpha\log\log k}\right)^k, \tag{2}$$

where α is any positive constant and k is large enough. As far as the lower bounds are concerned, limited progress seems to have been made since 1974 (see [1], [2], [4]). Infinite versions have also been studied in, for example, [7] and [9].

What appeals to us here is the similar problem for families having a fixed ground set. Define $F(n,r)$ to be the largest integer so that there exists a family $\mathcal{F}$ of subsets of an n-element set which does not contain a Δ-system of r sets. In [10], Erdős and Szemerédi showed

$$F(n,3) < 2^{n(1-\frac{1}{10\sqrt{n}})} \tag{3}$$

and stated that the probabilistic method implies that for each $r \geq 3$, there exists a constant $c_r > 0$, so that

$$F(n,r) > (1+c_r)^n$$

where $c_r \to 1$ as $r \to \infty$. Let

$$\beta_r = \lim_{n\to\infty} F(n,r)^{1/n}.$$

Abbott and Hanson [5] observed that β_r exists and that the probabilistic method mentioned above gives $\beta_r \geq 2(r+2)^{-1/r}$. They also presented a construction implying

$$\beta_r \geq \binom{2r-2}{r}^{1/(2r-2)} \sim 2^{(1-\frac{\log(2r)}{4r})}. \tag{4}$$

The Erdős-Szemerédi proof [10] of (3) reveals relations between bounds for $f(k,r)$ and $F(n,r)$. It shows that good upper bounds for $f(k,r)$ yield satisfactory upper bounds for $F(n,r)$ and strong lower bounds (if found) for $F(n,r)$ might imply lower bounds for $f(k,r)$. In Section 2, we repeat the Erdős-Szemerédi argument, however giving a more general outcome (Theorem 2.1) which yields the following two propositions.