A zoo of l_1-embeddable polytopal graphs*

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Abstract

A simple graph $G = (V, E)$ is called l_1-*graph* if, for some λ,$n \in \mathbf{N}$, there exists a vertex-addressing of each vertex v of G by a vertex $a(v)$ of the n-cube H_n preserving, up to the scale λ, the graph distance, i.e. $d_G(v, v') = \lambda d_{H_n}(a(v), a(v'))$ for all $v \in V$. We distinguish l_1-graphs between 1-skeletons of a variety of well known classes of polytopes: semiregular, regular-faced, zonotopes, Delaunay polytopes of dimension ≤ 4 and several generalizations of prisms and antiprisms.

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1 Introduction

A connected simple graph G is called an l_1-*graph* if its (shortest path) distance matrix d_G, seen as a metric space (d, X) on the set X of vertices of G, is embeddable isometrically into some l_1^m-space. The l_1^m-space is $\mathbf{R}^m$ with l_1-metric

$$d_{l_1}(x,y) = ||x-y||_{l_1} = \sum_{i=1}^{m} |x_i - y_i|.$$

Equivalently, see [AsDe80], G is an l_1-graph iff d_G is embedded isometrically into the restriction of l_1^n on an n-cube up to a certain *scale* λ . Clearly, $\lambda = 1$ (respectively, $\lambda = 2$) means that G is an isometric subgraph of the n-cube graph (respectively, of the n-half-cube graph). We denote these embeddings by $G \to H_n$ (respectively, $G \to \frac{1}{2}H_n$).

Any metric subspace A of l_1^m is (see [Dez60]) *hypermetric*, i.e. it satisfies

$$\sum_{1 \le i < j \le n} b_i b_j d_{l_1}(x_i, x_j) \le 0 \tag{1}$$

for every distinct $x_1, x_2, ..., x_n \in A$, $b := (b_1, ..., b_n) \in \mathbf{Z}^n$, and $\sum_{i=1}^n b_i = 1$. Clearly, in the special case $b = (1, 1, -1)$ the above inequality is the classical triangle inequality. Call the above inequality *5-gonal* (*2k+1-gonal*, respectively) if $b = (1, 1, 1, -1, -1)$ ($\sum_i |b_i| = 2k+1$, respectively). Call a graph G *hypermetric* (*5-gonal*, *(2k+1)-gonal*), if d_G is hypermetric (respectively, satisfies 5-gonal, the $(2k+1)$-gonal inequalities). Any $(2k+1)$-gonal graph is $(2k-1)$-gonal, [Dez60], but $K_{2k+1} - C_{k+1}$ is an example of a $(2k-1)$-gonal which is not $(2k+1)$-gonal for any $k \ge 2$. So, for a graph G the following chain of implications holds

$$(G \to H_{n_2}) \Rightarrow (G \to \frac{1}{2}H_{n_1}) \Rightarrow l_1\text{-graph} \Rightarrow \text{hypermetric} \Rightarrow (2k+1)\text{-gonal} \Rightarrow \text{5-gonal}$$

In this paper we systematically study well known polytopal graphs in connection within the above chain of notions, in particular, looking for l_1-graphs within 1-skeletons of polytopes with a degree of euclidean regularity. These polytopes include: regular-faced polyhedra, semi-regular polytopes, zonotopes, Delaunay polytopes of dimension ≤ 4, and some generalizations of prisms and antiprisms. In [DeSt96] one can find a related study of isometric, up to a scale, embeddings of 1-skeletons of *infinite* graphs (of Archimedean and Laves plane tilings) into *cubic lattices.*

Here are other instances of embeddings of graphs which are related to our subject:

1) well known cubical graphs (*induced* subgraphs of hypercubes) and a hypercube embedding with a *small distortion* (of distances) used in computer architecture.

2) isometric (or isometric with a small distortion) embeddings of graphs into other (than l_1-spaces) normed spaces; see [LLR95] and the huge bibliography there.

The following criterion for a graph G to be an l_1-graph is known, [Shp93], [DeGr93]: G is an l_1-graph iff it is an isometric subgraph of the direct product of half-cubes and Cocktail-Party graphs $K_{m\times 2}$ (1-skeleton of the cross-polytope β_m). Even, a good polynomial algorithm for a recognition of l_1-graphs is known [DeSp96]. So "raisons d'etre" for this paper are the following: