Limit sets of free groups, Hausdorff dimension and subshifts of finite type

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December 19, 1995

## §0. Introduction

Let $X$ be a locally compact separable metric space which has an infinite diameter. Assume that $X$ has a natural compact boundary $\partial X$ with a corresponding induced (cordal) metric. Suppose that $G$ is a discrete group of isometries of $X$ which acts properly discontinuously on $X$. Set

$$orb(O,G) = \{U : U = gO, g \in G\}, O \in X,$$
$$\Lambda(G,O) = closure(orb(O,G)) \cap \partial X.$$

The simplest invariants of $G$ are the Hausdorff dimension $\delta(G,O)$ of $\Lambda(G,O)$ and the critical exponent $\kappa(G)$ of the related Poincaré series. (We will show that $\kappa(G)$ is independent of the point $O \in X$.) A more sophisticated invariant is the family of Patterson-Sullivan measures on $\Lambda(G,O)$. See for example [**Pat**], [**Sul**] and [**Tuk**] for the case where $X = \mathbf{H}^n$ and $G$ is a discrete group of

hyperbolic isometries. In this case $\Lambda(G,O)$ and $\delta(G,O)$ do not depend on the choice of $O$. We will let $\Lambda(G) := \Lambda(G,O), \delta(G) := \delta(G,O)$ when these quantities do not depend on $O$.

Assume in addition that $F$ is a finitely generated free group on a set $S$ of $r$ generators. It is well known that $\Lambda(F,O)$ can be described as a standard subsfift of finite type (SFT) on $2r$ symbols - $\Gamma^\infty$ such that certain points of $\Lambda(F,O)$ have a multiple representation as points of $\Gamma^\infty$.

(If $F$ is a Fuchsian group then a countable number of points of the limit set have a double representation.) One can view $\Gamma^\infty$ as the boundary $\partial\mathcal{K}(F,S)$ of the infinite $2r$ regular tree $\mathcal{K}(F,S)$. (It is the tree induced by $orb(O,F)$.

In the Fuchsian case $\mathcal{K}(F,S)$ corresponds to the geometrical tiling induced by $S$.) Let $\phi$ be a metric on $\mathcal{K}(F,S)$. $\phi$ induces a metric $d_\phi$ on $\Gamma^\infty$ which makes $\Gamma^\infty$ a compact space (under some mild assumptions on $\phi$). Let $\delta(\phi)$ be the Hausdorff dimension of $\Gamma^\infty$ in $d_\phi$. There are two natural choices for $\phi$. The first one is the standard metric on an infinite tree by letting $a > 0$ be the distance between any two neighboring vertices. Denote by $d_a$ the induced metric on $\Gamma^\infty$. For this metric one has the simple equality $\delta(d_a) = \frac{1}{a}$. The second metric $\psi$ on $\mathcal{K}(F,S)$ is the metric induced by the original metric on $X$.

Let $\mathcal{E}$ be the set of probability measures on $\Gamma^\infty$ which are invariant and ergodic for the shift map. Choose $\mu \in \mathcal{E}$. In §5 we show that the metric $\psi$ on $\mathcal{K}(F,S)$ induced by the $X$-metric is essentially the natural metric $d_{\alpha(\mu)}$ for almost all vertices of $\mathcal{K}(F,S)$ with respect to $\mu$. The value $\alpha(\mu)$ is the limit of the "average" $X$-distance between the adjacent vertices of $\mathcal{K}(F,S)$. The existence of this limit is guaranteed by Kingman's subadditive ergodic theorem. Let $h(\mu)$ be the measure entropy of the shift map on $\Gamma^\infty$. Assume that $h(\mu) > 0$. We then show that

$$\kappa(F) \geq \frac{h(\mu)}{\alpha(\mu)}, \mu \in \mathcal{E}. \tag{0.1}$$

Moreover in the above inequality we may replace $\alpha(\mu)$ by $\frac{\alpha_m(\mu)}{m}, m = 1, ...,$ where $\alpha_m(\mu)$ is the $\mu$-average

$X$-distance of the vertices having graph distance $m$ to the origin $O$ (these vertices correspond to the words of length $m$ in $F$). These inequalities give a computable way to find lower bounds for $\kappa(F)$. In particular we have the inequality

$$\kappa(F) \geq \sup_{\mu \in \mathcal{E}, h(\mu) > 0} \frac{h(\mu)}{\alpha(\mu)}. \tag{0.2}$$

The works of Bowen [**Bow**] and Ruelle [**Rue**] suggest that in the case $F$ is Fuchsian, or more generally when $F$ is geometrically finite free group of hyperbolic isometries, then equality holds in (0.2). The maximal measure is the Bowen-Margulis-Ruelle measure which should be closely related to the Patterson-Sullivan measure.

Most of our paper is devoted to the various connections between the geometrical notions of groups as above and the corresponding notions in SFT. It seems that many results of these paper (including inequality (0.1-0.2)) can be extended to nonfree (Fuchsian) groups $G$ if the limit set $\Lambda(G, O)$ can be given by SFT. See for example [**Seri**] and the references therein. We are postponing this discussion to the near future.

We now survey briefly the content of our paper. §1 introduces the basic notions of the paper. We consider an infinite locally finite tree $\mathcal{T}$ with the standard metric. This metric induces a metric on the boundary $\partial\mathcal{T}$. Let $F$ be a finitely generated free group of isometries acting on $\mathcal{T}$. We then discuss the limit set $\Lambda(F, \mathcal{T})$. In §2 we consider the Hausdorff dimension $\delta(F, \mathcal{T})$ of $\Lambda(F, \mathcal{T})$. Assume that $G$ is a finitely generated subgroup of $F$. We then consider the problem when $\delta(G, \mathcal{T}) = \delta(F, \mathcal{T})$. We show that the above equality holds iff $G$ has a finite index in $F$. §3 considers the classical case of the modular group $PSL(2, \mathbf{Z})$. Let $\mathcal{T}$ be the Bruhat-Tits tree corresponding to the modular group. Assume that $G$ is a free finitely generated subgroup of the modular group. We then discuss the relations between the (standard) Hausdorff dimension $\delta(G)$ and the corresponding Hausdorff dimension $\delta(G, \mathcal{T})$. In §4 we show how a metric $\phi$ on an infinite locally finite tree $\mathcal{T}(\Gamma)$, induced by a given finite digraph $\Gamma$, gives rise to the metric $d_\phi$ on $\partial\mathcal{T}(\Gamma) = \Gamma^\infty$. In §5 we show that the metric $\phi$ on $\mathcal{T}(\Gamma)$ gives rise to a set of functions satisfying the conditions of Kingman's subadditive ergodic theorem. Denote by $\alpha(\mu) \geq 0$ the limit value of these functions with respect to an ergodic measure $\mu \in \mathcal{E}$. It then follows that the metric $\phi$ is essentially the natural metric $d_{\alpha(\mu)}$ on $\mathcal{T}(\Gamma)$. We then prove the inequalities (0.1-0.2). In the last section we give some arguments based on Ruelle's results in favor of the conjecture that for finitely generated free Fuchsian groups equality holds in (0.2). We then define and discuss an analog of the Patterson-Sullivan measure $\mu$ on $\Gamma^\infty$. We consider in detail the case of the modular group. Finally we show that for any group $G$ of discrete isometries containing a free subgroup on two generators and which acts freely and properly discontinuously on a locally compact separable matric space $\kappa(G) > 0$.

### §1. Limit sets of free groups acting on infinite trees

Let $\mathcal{T} = (V, E)$ be an infinite locally finite tree. Here $V$ is the set of vertices and $E \subset V \times V$ is a set of (undirected) edges. We suppose that $\mathcal{T}$ does not have leaves, i.e. the degree $deg(v)$ of any vertex $v \in V$ is greater than one. Introduce the standard metric on the set of vertices $V$ by letting

$$dg(u, u) := 0, u \in V;\ \ dg(u, v) := h > 0, u, v \in V, (u, v) \in E.$$

For any $u, v \in V, u \neq v$ set $dg(u, v) := kh$ where $k$ is the length of the unique path in $\mathcal{T}$ between $u$ and $v$. In what follows we assume that $h = 1$ unless stated otherwise.

Assume that $\mathcal{T}$ is an infinite locally finite tree with no leaves. The standard compactification of $\mathcal{T}$ is as follows. Fix a point $O$ (the origin). Then every point $a \in \partial\mathcal{T}$ is an infinite path $a = (v_i)_0^\infty, v_0 = O$, i.e. $v_{i+1} \neq v_{i-1}, i = 1, ...,$. Note that $a$ is a global geodesic. Since $\mathcal{T}$ is locally finite it follows that $\partial\mathcal{T}$ is a compact totally disconnected (Cantorian) set. We now define a usual metric on $\partial\mathcal{T}$. Fix $0 < \tau < 1$. (Later we show that there is a convenient choice of $\tau$ to make our formulas simpler.) For $a, b \in \partial\mathcal{T}$ we let $dist(a, b) = \tau^k$ where $k$ is the length of the maximal common subchain of $a$ and $b$ starting form $O$. We call the metric $dist(\cdot, \cdot)$ the *chordal* metric on $\partial\mathcal{T}$. It is possible to extend this metric to $\mathcal{T} \cup \partial\mathcal{T}$ as follows. Identify each $u \in V$ with a unique path $c = (v_i)_0^k, v_0 = O, v_k = u$. Let $d = (w_i)_0^m, 1 \leq m \leq \infty$ be either finite or infinite path starting from $O$. Then $dist(c, c) = 0$ and $dist(c, d) = \tau^k, c \neq d$

where $k$ is the length of the maximal common subchain of $c$ and $d$ starting from $O$. Let $dist'$ be the induced metric on $\mathcal{T} \cup \partial\mathcal{T}$ with respect the origin $O'$. It then follows that the corresponding metrics $dist, dist'$ are equivalent.

$$\frac{dist(a, b)}{K} \leq dist'(a, b) \leq K dist(a, b), a, b \in \mathcal{T} \cup \partial\mathcal{T},$$

for some $K \geq 1$.

Let $F$ be a finitely generated free group with a minimal set of generators $S = \{f_1, ..., f_r\}$. Then one associates with the pair $(F, S)$ the Cayley tree $\mathcal{K}(F, S) = (V, E), E = F$, where each vertex $v \in V$ is expressed as a reduced word

$$w = f_{i_1}^{\epsilon_1} \cdots f_{i_k}^{\epsilon_{i_k}}, \epsilon_j = \pm 1, j = 1, ..., k.$$