On nonnormal matrices that can be reduced to band form under finite sequences of elementary unitary similarities

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Abstract

We say that a square complex matrix is condensable if it can be reduced to a bandform through a finite sequence of elementary unitary similarities. The two main results of this paper are the following assertions:

(1) Any matrix with a low rank skew - Hermitian part is condensable.

(2) If A is a normal condensable matrix, then any rank one perturbation of A (generally nonnormal) is condensable as well.

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1 Introduction

In this paper we continue the investigation of the generalized Lanczos procedure introduced in [Elsner, Ikramov]. In the latter, mostly the case of a normal matrix has been studied. It has been shown that the application of the generalized Lanczos procedure to a matrix A produces a variable bandwidth matrix B that we call the condensed form of A. If A is normal the bandwidth is still variable but considerably smaller. For some important particular cases (including, of course, Hermitian and unitary matrices), the condensed form is a genuine band matrix with a modest bandwidth. Moreover, there exist situations when this band matrix, rather broad in its upper left corner, shrinks ultimately to the bandwidth 3.

Not all normal matrices can be reduced to a constant bandwidth form through finite sequences of unitary similarities. This is even more so for general nonnormal matrices. In fact, even if *infinite* sequences of unitary similarities are admitted, the following assertion can be proved by a simple dimension argument using some basic facts from algebraic geometry: "Given a fixed bandwidth p, then for $n \geq$ some integer depending on p, there are complex $n \times n$ matrices which are not unitarily similar to a p-banded matrix" (here the short paper [Sturmfels] was cited).

Nevertheless, there exist nonnormal matrices that can be unitarily reduced even to tridiagonal form. Some classes of such matrices can be found in [Longstaff]. Motivated by these results, we describe in this report several classes of nonnormal matrices that can be transformed, essentially through our generalized Lanczos procedure, into constant bandwidth matrices. Although similar in spirit to the results in [Longstaff], ours are different in at least three respects: firstly, we do not confine ourselves to the bandwidth three; furthermore, we do insist our condensed forms to be achieved via *finite* sequences of elementary unitary similarities; and, finally, they can be constructed with an arbitrary choice of the initial vector.

The paper is organized as follows. In Section 2, we fill a gap in the previous paper. There two versions of our generalized Lanczos procedure had been presented. Here we show the relations between them, a topic treated there in a more informal manner. In Section 3 linear dependencies inside generalized Krylow sequences are studied. The main results are in Section 4. It is shown (Theorem 6 and 7) that any matrix with low rank skew - Hermitian part can be reduced to bandform through a finite sequence of elementary unitary similarities. A similar result holds for rank one perturbations of normal matrices (Theorem 10).