Coordinates of maximal roots of weakly nonnegative unit forms

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1 Introduction and main result

1.1. A unit form is a map $\chi : \mathbb{Z}^n \to \mathbb{Z}$, $x = (x_1, \ldots, x_n) \mapsto \sum_{i=1}^{n} x_i^2 + \sum_{\substack{i,j=1 \\ i<j}}^{n} \chi_{ij} x_i x_j$ where $\chi_{ij} \in \mathbb{Z}$. Unit forms appear in the representation theory of finite-dimensional algebras and similar structures. They occur as Tits forms or Euler characteristics (see e.g. [Ga], [Bo], [Ri]) and their weak positivity resp. nonnegativity frequently is related to finite resp. tame representation type.

We recall the basic definitions concerning unit forms. The unit form χ is said to be weakly positive resp. nonnegative if $\chi(x) > 0$ resp. $\chi(x) \geq 0$ for all $x \in C_n$, $x \neq 0$ where C_n is the cone in $\mathbb{Z}^n$ consisting of all $x = (x_1, \ldots, x_n)$ with $x_i \geq 0$. The vectors $x \in C_n$, $x \neq 0$ are called positive. Obviously $\mathbb{Z}^n$ is partially ordered by $x \geq y$ if $x - y \in C_n$. The vectors $x \in \mathbb{Z}^n$, $x \neq 0$ satisfying $\chi(x) = 1$ resp. $\chi(x) = 0$ are called 1-roots resp. 0-roots of χ.

1.2. Let us now review more precisely the connection between the representations of an algebra A over an algebraically closed field k and the weak definiteness of its Tits form χ_A. To give the definition of χ_A, we suppose that A is basic, the ordinary quiver of A is directed and A has up to isomorphism exactly n simple modules $S_1, \ldots, S_n$. Then $\chi_A(x) = \sum_{i,j=1}^{n} (\sum_{\nu=0}^{2} (-1)^{\nu} \dim_k \mathrm{Ext}_A^{\nu}(S_i, S_j)) x_i x_j$. It is easy to see that this is a unit form in the above sense. The algebra A is called of finite representation type if there are only finitely many isomorphism classes of indecomposable finite-dimensional A-modules. In order to check finite representation type one may proceed as follows: Using covering theory (see [BG]) one may suppose that A is simply connected. This implies that A has a directed ordinary quiver and the Auslander-Reiten quiver of A has a postprojective component. For an algebra A of this kind it is shown in [Bo] that finite representation type is equivalent to the weak positivity of its Tits form χ_A.

But the connection between the representations of A and the form χ_A is even closer. Namely, the dimension vectors of the indecomposable A-modules are precisely the positive 1-roots of χ_A. Recall that the dimension vector $x = (x_1, \ldots, x_n)$ of an A-module X has as component x_i just the multiplicity of S_i in a composition series of X.

Concerning tame representation type the picture is not so complete. In [Pe1] it is shown that tame type of A implies the weak nonnegativity of the Tits form χ_A. Moreover there are interesting classes of tame algebras (see e.g. [Ri]) where the dimension vectors of the indecomposable A-modules are just the connected positive 1-roots and 0-roots of χ_A.

1.3. The above cited results show that informations about the possible coordinates x_i of positive 1-roots x of weakly positive unit forms χ are relevant at least for representation theory. A sharp bound for these coordinates is given by a theorem of Ovsienko ([Ov1], see also [Ri]) saying that always $x_i \leq 6$.

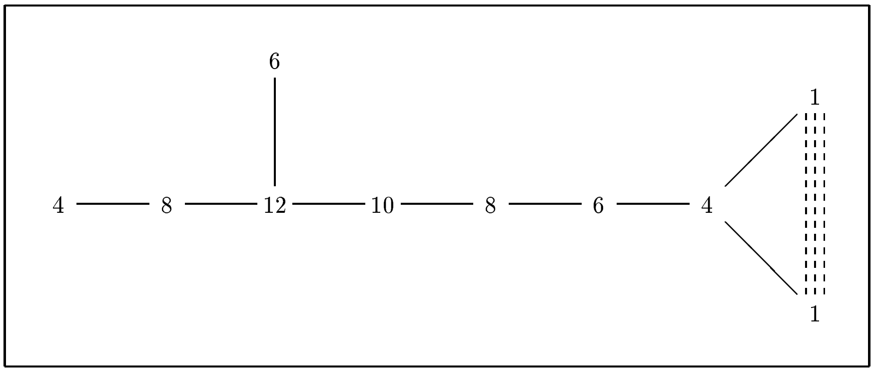
For weakly nonnegative forms the set of positive 1-roots is usually infinite. Hence it makes no sense to ask for bounds of the coordinates of positive 1-roots in general. On the other hand by a result due to Drozd (see e.g. [Ri]) weakly positive unit forms have only finitely many positive 1-roots. Using this fact, Ovsienko's theorem clearly may be reformulated as $x_i \leq 6$ for all maximal positive 1-roots x of a weakly positive unit form χ. This reformulation allows a reasonable generalization to weakly nonnegative forms which is actually the topic of this paper. Namely we want to answer the question whether there is a bound for the coordinates of the maximal positive 1-roots of weakly non-negative unit forms. Our main result is the following:

Main Theorem. *If $\chi : \mathbb{Z}^n \to \mathbb{Z}$ is a weakly nonnegative unit form and $x = (x_1, \ldots, x_n)$ is a maximal positive 1-root of χ, then $x_i \leq 12$ for all $i = 1, \ldots, n$.*

We will prove this theorem in section 9 after establishing several preliminary results some of which should be of interest on their own.

Obviously we can and will restrict to that case that x is sincere. In this case , the requirement that the unit form χ is weakly nonnegative is not really necessary but follows from the maximality . Namely, it is shown in [HP] under the assumption $\chi_{ij} \geq -5$ for all $i < j$ and in [Ov2] in general that the existence of a maximal sincere positive 1-root forces a unit form to be weakly nonnegative.

Let us present here an example χ where the bound 12 is actually reached. For displaying the form χ we use the usual attached bigraph which will be reviewed more detailed in 2.4. To the points of the bigraph χ we attach the components of the only sincere positive 1-root of χ.



1.4. The question may arise whether weakly nonnegative unit forms with maximal positive 1-roots appear in practice. But in fact the question for bounds of maximal positive