On the support of irreducible weight modules

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Abstract

For a reductive complex finite-dimensional Lie algebra L which does not contain E_8 as a direct summand we study the support of the irreducible weight modules. In particular we show that any such non-dense module is a quotient of generalized Verma module.

1 Introduction

Let L be a finite dimensional reductive Lie algebra over C, H be a Cartan subalgebra, Δ be the root system, π be a basis of Δ and Q be a free abelian group generated by π.

For an L-module, V, set $V_\lambda := \{v \in V |\ hv = \lambda(h)v, \text{ for all } h \in H\}$, where $\lambda \in H^*$. If V_λ is non-zero then λ is a *weight* of V and V_λ is the corresponding weight-space. We denote by $P(V)$ the set of all weights of V. An L-module is called weight if $V = \bigoplus V_\lambda$, $\lambda \in P(V)$. Clearly, if V is indecomposable module then $P(V) \subseteq \lambda + Q$, for some $\lambda \in H^*$.

A weight module V is said to be *torsion-free* if for all $\alpha \in \Delta$ the action of $X_\alpha : V \to V$ is injective, and to be *dense* if $P(V) = \lambda + Q$ for some $\lambda \in P(V)$. The classification of all irreducible torsion free sl(2)-modules is standard [B] and is obtained in [BFL] for the Lie algebra sl(3). The question remains open in other cases. Some classes of irreducible torsion free modules were studied in [BL].

Clearly, any torsion-free module is dense. The converse is also true for irreducible modules with finite-dimensional weight spaces [Fe], though it is not the case for modules with infinite-dimensional weight spaces [F].

It follows from [Fe] that any irreducible weight L-module with finite dimensional weight spaces is either torsion-free (in which case L is of type A_n or C_n) or the irreducible quotient of a generalized Verma module (GVM) induced from a torsion free module for parabolic subalgebra. This result is no longer valid for modules with infinite dimensional weight spaces. For such modules an analogous result for algebras of type A_n, B_n, C_n, D_n was obtained in [F].

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In particular, it was shown that irreducible weight module V for classical Lie algebras is either dense or is the quotient of GVM induced from a dense module for a parabolic subalgebra.

In this paper we re-examine the proof of Futorny's result for the classical Lie algebras and generalize it for any reductive algebra which does not contain E_8 as a direct summand.

We reduce the problem to some combinatorics on the root system (Lemma 3.3). In the case of algebras of type E_6, E_7 the needed computer software was developed. We present the result in the case of E_6 which can be checked by direct calculations. The case of E_7 is left out since it would take a lot of space even to sketch the result.

The structure of the paper is the following. In section 2 we recall the definition and basic properties of GVM 's and study the support of non-dense irreducible modules in section 3. The main result is Theorem 3.1 which provides a characterization of the support of a non-dense irreducible module. As a corollary we obtain that every such module is a unique irreducible quotient of a GVM .

2 Notation and definitions

We use [CF] as our main reference on GVM .

Let $\pi = \{\alpha_1, \dots, \alpha_n\}$, $\Delta^+ = \Delta^+(\pi)$ (respectively $\Delta^- = \Delta^-(\pi)$) be a set of all positive (resp. negative) roots in Δ with respect to π. Let S denotes an arbitrary subset of π; Q_S be a subgroup of Q generated by S; $\Delta_S := \Delta \cap Q_S$, ${}_+\mathcal{N}_S := \bigoplus L_\alpha$, ${}_-\mathcal{N}_S := \bigoplus L_{-\alpha}$ with $\alpha \in \Delta_S^+$; $\Delta_S^+ := \Delta^+ \cap Q_S$; Q_+^S be a monoid generated by $\Delta^+ \setminus \Delta_S$.

Let $\mathcal{H}^S := \{h \in \mathcal{H} |\ \alpha(h) = 0 \text{ for } \alpha \in S\}$ and $\mathcal{M}_S :={}_- \mathcal{N}_S \bigoplus \mathcal{H} \bigoplus_+ \mathcal{N}_S$. Further, let $\mathcal{N}_S^+ := \bigoplus L_\alpha$ and $\mathcal{N}_S^- := \bigoplus L_{-\alpha}$ where $\alpha \in \Delta^+ \setminus \Delta_S$. Then the subalgebra $\mathcal{B}_S := \mathcal{M}_S \bigoplus \mathcal{N}_S^+$ is a parabolic subalgebra of L with $\mathcal{M}_S$ as a Levi factor.

Let V be an irreducible weight $\mathcal{M}_S$-module. Consider V as a $\mathcal{B}_S$-module on which $\mathcal{N}_S^+$ acts trivially. An L-module

$$M(S,V) = \mathcal{U}(L) \otimes_{\mathcal{U}(\mathcal{B}_S)} V$$

is called *GVM* , associated with L, H, π, S, V [L,R].

DEFINITION 2.1 *Let V be a weight L-module and $S \subset \pi$, $\lambda \in P(V)$. An element $v \in V_\lambda$ is called S-semiprimitive (with respect to π) if $\mathcal{N}_S^+ v = 0$.*

LEMMA 2.2 *If V is irreducible L-module, $\xi \in Q_S$ and $v \in V_\lambda$ is S-semiprimitive element then any $v' \in V_\lambda$ is S-semiprimitive.*

Proof. Since V is irreducible then V_λ is irreducible module over the centralizer $C(H)$ of H in $\mathcal{U}(H)$ [D]. Thus, $v' = uv$ for some $u \in C(H)$ and $\mathcal{N}_S^+ v' = \mathcal{N}_S^+ uv = 0$ implying (ii). □

Let W (resp. W_S) be the Weyl group of Δ (resp. Δ_S). The main properties of GVM are collected in the following statement.