Combinatorics of Delaunay polytopes of the isodual lattice Q_{10}

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Abstract

The results of [9] are generalized and simplified for code lattices. As an example, the code lattice Q_{10}, mentioned and named in the paper [6], is considered. Q_{10} has two symmetric Delaunay polytopes P_5, P_3 and an asymmetric P'_5, and is generated by P_5. P_5 is a symmetrization of the cut polytope $PCut_5$, i.e. it is the convex hull of all cuts and their complements in the complete graph K_5. The cuts and their complements are all circuits of the regular matroid R_{10} [12]. Besides P_5 is the convex hull of the unique 10-dimensional closed odd system of 16 pairs of opposite vectors of norm 5 spanning equiangular lines at angle $arccos\frac{1}{5}$. P_5 is also the convex hull of all codewords of the linear binary code $C_{10} = [10, 5, 4]$. The second symmetric Delaunay polytopes P_3 is the convex hull of a system of 40 pairs of opposite vectors of norm 3 with $0, \pm 1$ inner products. The asymmetric Delaunay polytope P'_5 is the convex hull of an odd system of 32 vectors of norm 5 with $\pm 1, -3$ inner products.

9-dimensional facets of P_5 are convex hulls of vector representations of the well known strongly regular graphs: the 4×4 grid graph $L_2(4)$, the Shrikhande graph, the Petersen graph. Similarly, 4 regular subgraphs of the Shrikhande graph define 8-faces of P_5.

Lattices, graphs, odd systems and polytopal combinatorics interlace tightly in this study.

1 The lattice Q_{10}, the matroid R_{10}, the code [10,5,4], and the two-graph $\mathcal{T}(5, 10)$

A lattice is called *isodual* [6] if it is congruent to its dual. The lattice Q_{10} is an even isodual lattice of minimal norm 4. It is generated by indicator vectors of cuts and their complements in the complete graph K_5. Let $V_5 = \{1, 2, ..., 5\}$ and $E_5 = \{ij : 1 \leq i < j \leq 5\}$ be the sets of vertices and edges of K_5, respectively. For $S \subseteq V_5$, the *cut* $\delta(S) \subseteq E_5$ is the set of edges having exactly one vertex in the set S.

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Let $c(S)$ be the (10-dimensional) indicator vector of the cut $\delta(S)$, i.e. the vector of the space $\mathbf{R}^{E_5}$ such that

$$c_{ij}(S) = \begin{cases} 1 & \text{if } |\{ij\} \cap S| = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, $c(V_5 - S) = c(S)$. Let j_{10} be the all-one vector. Denote by $c^*(S) = j_{10} - c(S)$ the indicator vector of the complement $\delta^*(S) = E_5 - \delta(S)$ of the cut $\delta(S)$.

Let $S\Delta T = (S - S \cap T) \cup (T - S \cap T)$ be the symmetric difference of the sets S and T. Using the identity $\delta^*(S) = E_5 \Delta \delta(S)$, it is easy to verify that

$$\delta(S\Delta T) = \delta(S)\Delta\delta(T) = \delta^*(S)\Delta\delta^*(T), \ \ \delta^*(S\Delta T) = \delta^*(S)\Delta\delta(T) = \delta(S)\Delta\delta^*(T). \qquad (1)$$

The 30 nontrivial sets $\delta(S)$, $\delta^*(S)$ for $S \subseteq V_4$, $S \neq \emptyset$, form the family of cycles of the famous regular matroid R_{10} of rank 5 (see, for example [12]). Cocycles of R_{10}, i.e. cycles of the dual matroid R^*_{10}, are all minimal by inclusions subsets of E_5 having intersections of even cardinality with all cycles of R_{10}. These are all 15 circuits of length 4 of the graph K_5 and all 15 their complements. There is a permutation of the edge-set E_5 determining an isomorphism between the sets of all cycles and all cocycles of R_{10}. For example, let $m \in V_5$ and $\{ijkl\} = V_5 - \{m\}$. Let (ij, jk, kl, il), (ij, jl, kl, ik), (ik, jk, jl, il) be the three 4-circuits of K_5 on the 4-set $\{ijkl\}$. Then there is a permutation of E_5 such that these circuits are mapped into $\delta(\{m\}) = \{im, jm, km, lm\}$, $\delta^*(\{ij\}) = \{ij, kl, km, lm\}$, $\delta^*(\{kl\}) = \{ij, kl, im, jm\}$. So we have that R_{10} is permutationally equivalent (i.e. isomorphic) to R^*_{10} (but not self-dual). The automorphism group of R_{10} is doubly transitive [12].

Since the matroid R_{10} is a regular matroid of rank 5, it has a representation by a 5-dimensional subspace of the 10-dimensional binary space of the indicator vectors of all subset of the 10-set E_5. The vectors $c(S)$, $c^*(S)$, $S \subseteq V_4$, provide one of the representations of R_{10}.

Recall that each k-subspace of a binary n-space $\mathbf{B}^n$ forms a binary linear code $[n, k, d]$, where d is the minimal length of code words. Hence the 32 vectors $c(S)$ and $c^*(S)$ form the code $C_{10} = [10, 5, 4]$. This code has 1 word $c(\emptyset) = 0$ of length 0, 15 words of length 4, 15 words of length 6 and 1 word $c^*(\emptyset) = j_{10}$ of length 10. Similarly to the matroid R_{10}, this code is isomorphic to its dual, although is not self-dual.

The lattice Q_{10} is obtained from the code C_{10} by Construction A of [3]. This means that every lattice vector

$a = \sum_S (z_S c(S) + z'_S c^*(S)) \in Q_{10}$ with $z_S, z'_S \in \mathbf{Z}$ has the form

$$a = c(S) + 2f \text{ or } a = c^*(S) + 2f$$

for some S, where $f \in \mathbf{Z}^{10}$ is an integral vector. Since the code C_{10} is isomorphic to its dual, the dual lattice $Q^*_{10} = \{d \in \mathbf{R}^{E_5} : da \in \mathbf{Z}, a \in Q_{10}\}$ consists of the vectors $\frac{1}{2}b + f$, where b is a word of the dual code and $f \in \mathbf{Z}^{10}$. Hence Q^*_{10} is isomorphic to $\frac{1}{2}Q_{10}$, i.e. Q_{10} and Q^*_{10} are congruent, i.e. Q_{10} is isodual.