Stochastic quantization of the two-dimensional polymer measure

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Abstract

We prove that there exists a diffusion process whose invariant measure is the two-dimensional polymer measure ν_g. The diffusion is constructed by means of the theory of Dirichlet forms on infinite-dimensional state spaces. We prove the closability of the appropriate pre-Dirichlet form which is of gradient type, using a general closability result by two of the authors. This result does not require an integration by parts formula (which does not hold for the two-dimensional polymer measure ν_g) but requires the quasi-invariance of ν_g along a basis of vectors in the classical Cameron-Martin space such that the Radon-Nikodym derivatives (have versions which) form a continuous process. We also show the Dirichlet form to be irreducible or equivalently that the diffusion process is ergodic under time translations.

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1 Introduction and main results

Let ν be a given probability measure on a space X. The stochastic quantization of ν means the construction of a Markov process whose invariant measure is just ν. Parisi and Wu [PW81] first studied the stochastic quantization for certain gauge fields in quantum field theory. After that, many works (see e.g. [J-LM85], [AR89a], [AR89b], [AR91]) on the stochastic quantization of Euclidean quantum fields such as the $P(\phi)_2$ - field appeared. There is a close connection between the Edward's model (or say polymer measure) and the ϕ^4 - field (see e.g. [Sy69, AFH-KL86]). This naturally directs the interest to the stochastic quantization of a polymer measure. The main aim of the present paper is to study the stochastic quantization of the two dimensional polymer measure.

Let us first introduce the (rigorous) definition of the two dimensional polymer measure ν_g. Let $X := C([0,1], \mathbb{R}^2)_0$ be the set of all continuous paths in $\mathbb{R}^2$ indexed by $[0,1]$ and starting at zero. Let $\mathcal{B}$ be the σ-algebra generated by all maps $\omega \mapsto \omega_t,\ t \geq 0$, from X to $\mathbb{R}^2$ and let ν_0 denote the Wiener measure on $(X, \mathcal{B})$. Let $\alpha(x, A)$ be the self-intersection local time at $x \in \mathbb{R}^2$ of $\omega \in X$ on the set $A \subset [0,1] \times [0,1]$, i.e.

$$\alpha(x, A) = \int_A \delta_x(\omega_s - \omega_t) ds dt.$$

(For its precise definition, the reader is referred to [Ro86] and the references therein.) It is well-known that $\alpha(0, \{(s,t) : 0 \leq s < t \leq 1\}) = \infty,\ \nu_0 - a.e.$ Therefore, one has to use renormalization. To this end we set

$$\alpha_{i,k} = \alpha\Big(0, \Big[\frac{2(i-1)}{2^k}, \frac{2i-1}{2^k}\Big) \times \Big[\frac{2i-1}{2^k}, \frac{2i}{2^k}\Big)\Big),\ \ i = 1, \dots, 2^{k-1},\ k = 1, 2, \dots.$$

Let E be the expectation with respect to ν_0, and

$$\xi_n = \sum_{k=1}^{n} \sum_{i=1}^{2^{k-1}} (\alpha_{i,k} - E\alpha_{i,k}), \quad n \geq 1.$$

Then one can prove (see [LG85], [V69]) that $(\xi_n)_{n\geq 0}$ is almost surely convergent to a random variable $\xi \in L^2(\Omega; \nu_0)$ and $\lim_{n\to\infty} E|\xi_n - \xi|^2 = 0$. The random variable ξ is usually called the normalized self-intersection local time of planar Brownian motion. It is proved in [AHZ95] that the random variable ξ is not differentiable in the sense of Meyer-Watanabe. One can prove that there is $g_0 \in (0, \infty)$ (see e.g. [LG94], [PY96]) such that

$$E \exp(-g\xi) \begin{cases} < \infty, & \forall g \in (-g_0, \infty), \\ = \infty, & \forall g \in (-\infty, -g_0). \end{cases}$$

The two dimensional polymer measure ν_g is defined by

$$\nu_g := (E \exp(-g\xi))^{-1} \exp(-g\xi)\ \nu_0, \quad g \in (-g_0, \infty).$$