Stochastic quantization of the three-dimensional polymer measure

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Abstract

We prove that there exists a diffusion process whose invariant measure is the three-dimensional polymer measure ν_λ. The diffusion is constructed by means of the theory of Dirichlet forms on infinite-dimensional state spaces. We prove the closability of the appropriate pre-Dirichlet form which is of gradient type, using a general closability result by two of the authors. This result does not require an integration by parts formula (which does not even hold for the two-dimensional polymer measure ν_λ) but requires the quasi-invariance of ν_λ along a basis of vectors in the classical Cameron-Martin space such that the Radon-Nikodym derivatives (have versions which) form a continuous process.

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1 Introduction

In [AHRZ95] the stochastic quantization of the two-dimensional polymer measure was constructed using the theory of Dirichlet forms on infinite-dimensional state spaces. In this paper the same is done in the case of the three-dimensional polymer measure which is technically much more difficult since in three dimensions the polymer measure is no longer absolutely continuous w.r.t. Wiener measure. For the motivation and the background literature on stochastic quantization and the polymer measure (sometimes also called Edward's model) we refer to [AHRZ95, Section 1] and also to [W80, W82].

In order to describe our results more precisely we start with recalling some facts about the three-dimensional polymer measure. The polymer measure is informally defined by

$$\nu_\lambda(d\omega) = N_\lambda^{-1} \exp\Big(-\lambda \int_0^1 \int_s^1 \delta(\omega_t - \omega_t) ds dt \Big) \nu_0(d\omega), \tag{1.1}$$

where $\lambda \in [0, \infty)$ is the coupling constant, N_λ is a renormalization constant to assure that ν_λ is a probability measure, ν_0 is Wiener measure on $X := C_0([0,1] \to \mathbb{R}^3)$ and $\omega_t \mapsto \omega$, $t \geq 0$, the evaluation maps from X to $\mathbb{R}^3$. The three dimensional polymer measure was first constructed by J. Westwater (see [W80] and [W82]). A simpler construction for small values of the coupling constant was recently given by E. Bolthausen [B93]. In fact, the latter approach is still suitable for all positive coupling constants, and the polymer measure given by Bolthausen is actually the same as the one given by Westwater (see the recent work by Albeverio and Zhou [AZ95]). It was shown in [W82] that the three-dimensional polymer measure is singular with respect to Wiener measure. A rigorous definition of ν_λ can be given as follows (cf. [B93], [AZ95]). For the rest of this paper we fix $\lambda \in [0, \infty)$. For $\varepsilon \in (0, 1)$ we set $\kappa_1(\varepsilon) := 2(2\pi)^{-3/2}(\varepsilon^{-1/2} - 1)$, and

$$\kappa_2(\varepsilon) := \int_0^1 \int_{s_1 \vee \varepsilon}^1 \int_{s_2 \vee (s_1+\varepsilon)}^1 (2\pi)^{-3} (s_1(s_2-s_1) + s_1(s_3-s_2) + (s_2-s_1)(s_3-s_2))^{-3/2} ds_1 ds_2 ds_3.$$

It is easy to check that $\lim_{\varepsilon \to 0+} (2\pi)^2 \kappa_2(\varepsilon) |\log \varepsilon|^{-1} = 1$. Let

$$\begin{aligned} J^\varepsilon_{s,t;u,v} &:= \int_s^t \int_{(u \vee (\sigma+\varepsilon)) \wedge v}^v \delta(\omega_\sigma - \omega_\tau) d\sigma d\tau, \\ \bar{J}^{\varepsilon,\lambda}_{s,t} &:= \lambda J^\varepsilon_{s,t} - \lambda(t-s)\kappa_1(\varepsilon) + \lambda^2 (t-s) \kappa_2(\varepsilon), \end{aligned}$$

where $J^\varepsilon_{s,t} := J^\varepsilon_{s,t;s,t}$ (the functional $J^\varepsilon_{s,t;u,v}$ was already discussed in [Ro83] where the reader also finds the precise definition of the double integral). Let

$$\mu_{\varepsilon,\lambda} := (E \exp(-\bar{J}^{\varepsilon,\lambda}_{0,1}))^{-1} \exp(-\bar{J}^{\varepsilon,\lambda}_{0,1})\ \nu_0,$$

where E is the expectation with respect to ν_0. It was shown in [B93] and [AZ95] that the limit $\lim_{\varepsilon \to 0+} \mu_{\varepsilon,\lambda}$ exists in the weak sense. This limit is denoted by ν_λ. ν_λ is obviously a rigorous version of (1.1).

As already emphasized in [AHRZ95] the construction of the stochastic quantization of ν_λ by the theory of Dirichlet forms uses ν_λ and is based on some of its specific properties.