Combinatorial classification of finitely generated virtually free groups

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1. Introduction

For a finitely generated virtually free group Γ denote by m_Γ the least common multiple of the orders of the finite subgroups in Γ and let

$$b_\Gamma(\lambda) = \text{ number of free subgroups of index } \lambda m_\Gamma \text{ in } \Gamma\,.$$

Growth behaviour and asymptotics of this function $b_\Gamma(\lambda)$ have been investigated in two previous papers [M1] and [M2]. In [M1] we also introduced an equivalence relation $\sim$ on the class $\mathcal{C}$ of all finitely generated virtually free groups by setting

$$\Gamma_1 \sim \Gamma_2 :\Leftrightarrow (m_{\Gamma_1}, b_{\Gamma_1}) = (m_{\Gamma_2}, b_{\Gamma_2})\,,$$

i.e. two groups Γ_1 and Γ_2 are identified via $\sim$ if and only if Γ_1 and Γ_2 contain the same number of free subgroups for each given finite index. We gave a characterization of this equivalence in terms of a certain structural invariant, the *type* $\tau(\Gamma)$ of Γ, and we showed that each $\sim$–class decomposes into finitely many isomorphism classes. Hence, one can say that a group Γ is "almost determined" by the combinatorial datum (m_Γ, b_Γ).

The aim of the present note is to introduce a coarser equivalence relation $\sim_a$ into this context, which classifies groups by the asymptotic behaviour of the number of free subgroups of given finite index, and to characterize this equivalence $\sim_a$ again in terms of structural invariants. We show that the $\sim_a$–class of a group $\Gamma \cong \pi_1(\Gamma(-), Y)$ is determined by the constant m_Γ, the Euler characteristic $\chi(\Gamma)$ of Γ, the rank of the fundamental group $\pi_1(Y)$ of the graph Y underlying the above Stallings decomposition of Γ and the values of certain sums formed from the type $\tau(\Gamma)$. This characterization allows us to strengthen the finiteness result [M1, Theorem 3] on $\sim$–classes by showing that even a $\sim_a$–class contains only a finite number of non–isomorphic groups and we provide examples of virtually free groups Γ_1 and Γ_2 such that $\Gamma_1 \sim_a \Gamma_2$ and $\Gamma_1 \not\sim \Gamma_2$. These results contrast in an interesting way with our recent results concerning the function $s_\Gamma(n)$ counting the number of *all* subgroups of index n in Γ.

2. The type $\tau(\Gamma)$ and the equivalence relation $\sim_a$

Recall that, as a well-known consequence of Stallings' structure theorem, a finitely generated virtually free group Γ can be presented as the fundamental group of a finite graph of groups $(\Gamma(-), Y)$ with finite vertex groups $\Gamma(v)$; cf. [KPS] or [D, IV.1.9]. The fact that, conversely, the fundamental group of a finite graph of finite groups is always virtually free of finite rank is more elementary; see [Se, II.2.6, Prop. 11]. Fix such a decomposition $\Gamma \cong \pi_1(\Gamma(-), Y)$ of Γ and denote by V, E the set of vertices and (geometric) edges of Y, respectively. The type $\tau(\Gamma)$ of Γ is defined as the tuple $(m_\Gamma; \zeta_1(\Gamma), \ldots, \zeta_\kappa(\Gamma), \ldots, \zeta_{m_\Gamma}(\Gamma))$, where the $\zeta_\kappa(\Gamma)$ are integers indexed by the divisors of m_Γ, given by

$$\zeta_\kappa(\Gamma) = \left|\left\{e \in E : |\Gamma(e)| \,\Big|\, \kappa\right\}\right| - \left|\left\{v \in V : |\Gamma(v)| \,\Big|\, \kappa\right\}\right|.$$