UNIFORM DOMAINS OF HIGHER ORDER III

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Abstract. We continue the study of (p, c)-uniform domains. Special emphasis is on (p, c)-NUD sets. The ambient space is assumed to be a Hilbert space.

1. Introduction

This paper is a continuation to [Al] and [AV], and we assume that the reader is familiar with these papers. They will be cited as I and II. For example, I.6.11 means the result 6.11 of [Al]. For basic notation and terminology, see I, p. 6, and II.1.4. However, we assume throughout this paper that the space E is *a Hilbert space* instead of a general normed space. This simplifies several proofs and also gives better estimates for various constants. The finite-dimensional case $E = R^n$ is probably the most interesting.

The main emphasis will be on the null-sets (NUD) for homotopically (htop) and homologically (hlog) (p, c)-uniform domains in E. In Section 2 we give elementary estimates for the uniformity constants of certain standard domains. Section 3 deals with cartesian products of NUD sets. The case where the set is contained in a linear subspace of E is studied in Section 4. In Section 5 we consider compact sets in infinite-dimensional spaces and an application of our theory to bilipschitz spheres.

We let $x \cdot y$ denote the inner product of two vectors $x, y \in E$. For $A \subset E$ and $r > 0$ we write

$$\overline{B}(A, r) = \{x : d(x, A) \leq r\},\ S(A, r) = \{x : d(x, A) = r\}.$$

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2. Estimates for uniformity constants

2.1. *Summary of Section* 2. We give estimates for the uniformity constants of various standard domains. All proofs are elementary. Remember that E is a Hilbert space. If E is infinite-dimensional, Theorem 5.2 will give in certain cases constants better than the ones obtained in this section.

2.2. **Lemma.** *Each bounded set $A \subset E$ is contained in a ball $\overline{B}(a,r)$ with $r = d(A)/\sqrt{2}$.*

Proof. If $\dim E = n < \infty$, this follows from Jung's theorem [Fe, 2.10.41]. Indeed, one can choose $r = d(A)\sqrt{n/2(n+1)}$. The infinite-dimensional case was proved by J. Daneš [Da, Th. 2]. □

2.3. **Theorem.** *Each ball and half space in E is htop and hlog (p,c)-uniform for all $0 \le p < \dim E - 1$ with $c = \sqrt{3/2} = 1.22474\ldots$*

Proof. Suppose first that G is a ball, which can be assumed to be the unit ball $B(1)$. We show that G is htop (p,c)-uniform. Let $f : S^p \to G$ be continuous with $d = d(|f|) > 0$. If $d \ge 1$, we let $g : \overline{B}^{p+1} \to E$ be the cone extension of f with $g(0) = 0$. The uniformity conditions are clearly satisfied with the constant 1. Assume that $d < 1$. By 2.2, there is a ball $B = \overline{B}(a, d/\sqrt{2})$ containing $|f|$. We may choose a so that $|a| \le 1$.

Case 1. $B \subset G$. Let $g : \overline{B}^{p+1} \to E$ be the cone extension of f with $g(0) = a$. Each pair of points in $|g|$ lies in a triangle with side lengths at most d, and hence $d(|g|) \le d(|f|)$. If $x \in |g|$, then $x \in [a,y]$ for some $y \in |f|$, and we get

$$d(x, |f|) \le |x - y| \le d(x, \partial B) \le d(x, \partial G).$$

Again, the uniformity conditions hold with the constant 1.

Case 2. $B \not\subset G$. Choose $e \in \partial G$ with $a \in [0,e]$, and set $b = (1-d)e$. Then $|b| = 1 - d < |a|$ and $B \cap G \subset \overline{B}(b, cd)$ with $c = \sqrt{3/2}$. Let $g : \overline{B}^{p+1} \to E$ be the cone extension of f with $g(0) = b$. Each pair of points in $|g|$ lies in a triangle with side lengths at most cd, and hence $d(|g|) \le cd$.

To prove the lens condition, let $x \in |g|$. Then $x = (1-t)y + tb$ for some $y \in |f|$ and $0 \le t \le 1$. We have

$$|x| \le (1-t)|y| + t|b| \le 1 - t + t(1-d) = 1 - td,$$

and hence $d(x, G) = 1 - |x| \ge td$. This implies that

$$d(x, |f|) \le |x - y| = t|y - b| \le tdc \le cd(x, \partial G),$$

which is the lens condition.

Since the proof made use of solely cone extensions, it is valid with obvious modifications also in the homological case. The case where G is a half space follows from the above, since each compact set in G is contained in a ball $B \subset G$. □