WHEN IS THE STUDENT t–STATISTIC ASYMPTOTICALLY STANDARD NORMAL?

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Abstract

Let X, X_i, $i \in \mathbf{N}$, be independent identically distributed random variables. It is shown that the Student t–statistic based upon the sample $\{X_i\}_{i=1}^n$ is asymptotically $N(0,1)$ if and only if X is in the domain of attraction of the normal law. It is also shown that, for any X, if the self–normalized sums $U_n := \sum_{i=1}^n X_i \Big/ \left(\sum_{i=1}^n X_i^2\right)^{1/2}$, $n \in \mathbf{N}$, are stochastically bounded then they are uniformly subgaussian that is, $\sup_n \mathbf{E} \exp(\lambda U_n^2) < \infty$ for some $\lambda > 0$.

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1. Introduction. Let $X, X_i,\ i \in \mathbf{N}$, be independent, identically distributed random variables and let

$$S_n = \sum_{i=1}^{n} X_i, \quad V_n^2 = \sum_{i=1}^{n} X_i^2, \quad n \in \mathbf{N}. \tag{1.1}$$

Using this notation we can write the classical Student t–statistic as

$$T_n \ = \ \frac{S_n/V_n}{\sqrt{(n-(S_n/V_n)^2)/(n-1)}}. \tag{1.2}$$

The main object of this article is to prove that T_n or, equivalently, the self–normalized sum S_n/V_n, is asymptotically standard normal if and only if X is in the domain of attraction of the normal law, written $X \in DAN$, and $\mathbf{E}X = 0$.

A basic requirement in the proof of this result is that the moments of S_n/V_n converge to those of the standard normal whenever S_n/V_n is asymptotically $N(0,1)$. This is a trivial consequence of our second main result, which states that, if the sequence $\{S_n/V_n\}$ is stochastically bounded, then it is uniformly subgaussian in the sense that $\sup_n \mathbf{E}\exp\{\lambda(S_n/V_n)^2\} < \infty$ for some $\lambda > 0$ or, equivalently, $\sup_n \mathbf{E}\exp\{tS_n/V_n\} \leq 2\exp\{ct^2\}$ for all $t \in \mathbf{R}$ and some $c < \infty$. To compare this with previous results, we recall that Logan et *al.* (1973) show, among other things, that if X is in the domain of attraction of an α–stable law, $0 < \alpha \leq 2$, centered if $\alpha > 1$ and symmetric if $\alpha = 1$, then the sequence of self–normalized sums converges in distribution to a limit which is subgaussian, and if moreover X is symmetric, then the moments of S_n/V_n also converge to the corresponding moments of the limit. As a consequence of our second result we obtain that if the sequence $\{S_n/V_n\}$ is stochastically bounded then all its subsequential limits in law are (uniformly) subgaussian and there is convergence of the moment generating function of S_n/V_n along each convergent (in distribution) subsequence to the moment generating function of the limit, for all $t \in \mathbf{R}$ (there is even convergence of some square exponential moments).

If X is symmetric then the self–normalized sums are always stochastically bounded as a consequence of Khinchin's inequality. But if X is not symmetric, there are even centered random variables with good moment conditions for which S_n/V_n fails to be shift tight. We present examples of this type and give as well a criterion for stochastic boundednesss of self–normalized sums.

The line of research leading to our results starts perhaps with Efron (1969), who studied the limiting behavior of the Student t–statistic and, equivalently, of self–normalized sums in some nonstandard cases. In a more strict sense, it actually begins with the conjecture of Logan et *al.* (1973) stating that 'S_n/V_n is asymptotically normal if [and perhaps only if] X is in the domain of attraction of the normal law' (and X is centered).

The 'if' part of this conjecture follows rather easily from basic principles (Raikov's theorem), as was noticed, among others, by Maller (1981), but the parenthetical 'only if' part has remained open until now for the general case of not necessarily symmetric random variables. For X symmetric, Griffin and Mason (1991) attribute to Roy Erickson a beautiful proof of the fact that convergence in distribution of the self–normalized sums S_n/V_n to the standard normal law does indeed imply that $X \in DAN$. This result and its method of proof directly inspired ours.