Lower Bounds for the Spectral Gap of Markov chains

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Abstract. Well-known inequalities for the spectral gap of a Markov chain, like Poincaré's and Cheeger's Inequality, do not perform well if the transition graph of the Markov chain is strongly connected. Using the coupling methods we derive a lower bound for the spectral gap of a discrete-time Markov chain. Another lower bound is derived via hitting times. Finally the bounds are discussed using nearest-neighbor random walks on the circle, star and d-dimensional cube, for which the inequalities mentioned above can be considerably improved.

Keywords: Coupling, Markov chain, Spectral Gap

1 Introduction and Results

Let $(M, \mathcal{M})$ be a measurable space. We will denote by $\mathcal{M}_b$ the set of $\mathcal{M}$-measurable and bounded functions $f : M \to \mathbb{R}$. Any transition kernel Q from $(M, \mathcal{M})$ to a measurable space $(N, \mathcal{N})$ determines a linear operator from $\mathcal{M}_b$ to $\mathcal{N}_b$ by the definition

$$[Qf](x) = \int_N f(y)Q(x, dy), \; x \in M. \tag{1}$$

Moreover we have $\|Qf\| \leq \|f\|$, where $\|f\| = \sup\{ f(x) \mid x \in M \}$ denotes the supremum norm.
In our applications we choose $M = N$ and M to be a finite set. If Q is time-reversible with respect to the probability measure π the operator $[Q\cdot]$ defined in (1) is self-adjoint, so that all the eigenvalues of Q are real. By the assumption of irreducibility the constant eigenfunctions are the only eigenfunctions for the eigenvalue 1 (compare [SE]). Therefore we can enumerate the eigenvalues in the form

$$1 = \beta_0 > \beta_1 \geq \beta_2 \geq \ldots \geq \beta_{|M|-1} \geq -1.$$

The last inequality is strict if and only if the chain Q is aperiodic.

Let $\Omega = S^{\mathbb{N}}$, where $(S, \mathcal{S})$ is a measurable space and denote by $Y_n(\omega) = \omega_n$ the coordinate process. We assume that there exists a stationary measure $\mu \neq 0$ for the transition kernel Q. Recall that a stationary measure for countable infinite S in general does not exist any more, (compare page 54 in [JA86]). Denote by $T_y = \inf\{\, n \geq 0 \,|\, Y_n = y \,\}$ the first hitting time of the point y. As a lower bound for the spectral gap we get

Proposition 1.1 *(Hitting Bound) Let $f \in \mathcal{S}_b$ be an eigenvector for $\beta \neq 1$ of the transition kernel Q on $(S, \mathcal{S})$ and suppose that $E_x(T_y) < \infty$ for all $x, y \in S$. Then*

$$|1 - \beta| \geq \frac{1}{\sup_{x,y \in S} E_x(T_y)}. \tag{2}$$

As we will see in the applications the hitting bound performs well for the nearest-neighbor random walk on the circle, while it is not precise for nearest-neighbor random walk on the star and the cube. The proof of Proposition 1.1 is based on a martingale argument, that is similar to that used in the proof the coupling bound.

Recall the definition of a coupling:

Definition 1.2 *(Coupling of probability measures).*
Let $(\Omega_i, \mathcal{A}_i, P_i)$ for $i \in \{1, 2\}$ be probability spaces. A probability measure $\widetilde{P}$ on

$$(\Omega, \mathcal{A}) := (\Omega_1 \times \Omega_2, \mathcal{A}_1 \otimes \mathcal{A}_2)$$

is called a coupling of (P_1, P_2), if the following marginal conditions are satisfied:
i) $\widetilde{P}(A_1 \times \Omega_2)$=$P_1(A_1)$ *for all* $A_1 \in \mathcal{A}_1$
ii) $\widetilde{P}(\Omega_1 \times A_2)$=$P_2(A_2)$ *for all* $A_2 \in \mathcal{A}_2$.

Let $\Omega_1 = \Omega_2 = S^{\mathbb{N}}$ for $(S, \mathcal{S})$ be a measurable space and $Q : S \times \mathcal{S} \to [0, 1]$ be a transition kernel on S. The coupling measure is defined on $(\Omega, \mathcal{A}) := ((S^2)^{\mathbb{N}_0}, \otimes_{k=0}^{\infty} \mathcal{S}^2)$ and the coordinate process is denoted by $Y_n(\omega) = (Y_n^1(\omega), Y_n^2(\omega)) = (\omega_n^1, \omega_n^2)$, $\omega \in \Omega_1 \times \Omega_2$. Let T be the hitting time for the diagonal in $S \times S$, e.g.

$$T = \inf\{\ n \geq 0 \ \mid\ Y_n^1 = Y_n^2\ \}.$$

In the sequel we will be interested in measures $\widetilde{P}$ on the sequence space $(\Omega, \mathcal{A})$, such that a given sequence $(\widetilde{Z}_n)_n$ of random variables is a martingale. For special coupling measures this martingale property holds. In order to be able to specify these special coupling measures we introduce

Definition 1.3 *(Coupling of transition kernels). Let*

$$Q_i : \Omega_i \times \mathcal{A}_i \to [0, 1] \quad i \in \{1, 2\}$$