RESTORATION OF BINARY IMAGES FOR BERNOULLIAN NOISE MODELS

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December 11 1996

Abstract. In this paper the problem of the restoration of binary images from Bernoulli random noise is investigated. Assuming that an image has regular boundary, we show that linear averaging estimators are nearly optimal up to a logarithmic factor of the number of pixels. Moreover, we provide examples showing that for any estimator there exist images with regular boundary such that an estimation error is of the order of the size of this boundary. Furthermore we introduce a class of *hierarchical threshold* estimators which are wavelet based estimators achieving this lower bound. In terms of Besov norms these estimators have optimal convergence rates. Some simulation examples illustrate the performance of these estimators.

1. Introduction.

The restoration of images from a random noise is an important problem in image processing and non-parametric estimation. Various methods have been proposed for this purpose. One can divide them in two classes. Methods which use specific prior information and methods which use generic prior information like a specification of smoothness, e.g., Besov norms. Below we consider algorithms for the latter. Extensive numerical testing of several estimation procedures which do not use specific prior information and are based on linear averaging, non-linear thresholding and morphology operations, led to the conclusion that all of them have a comparable quality of restoration of binary images. To explain this observation in a model which is comparatively simple to analyze we investigate in Section 2 the problem of recovering binary images sampled with Bernoulli random noise as studied in a number of papers, see for instance, Greig *et al.* (1989). We prove that for a set of images with "simple" boundary (for instance, areas restricted by differentiable curves or polygons) linear averaging estimators are optimal in terms of the L^s-risk, $s \geq 1$, (or equivalently in

[1]Supported by the SFB 343 in Bielefeld

1991 *Mathematics Subject Classification.* Primary 62G05; secondary 62M40.

Key words and phrases. Binary images, random noise, L^s-losses, L^s-risk, linear estimator, threshold estimator.

the number of misclassified pixels) up to a logarithmic factor. See Corollary 2.3 and Theorem 2.4. But the situation may change drastically once specific prior information is available. In Example 2.6 we show that by using such a prior information one can restore a rectangle by a simple linear estimator with exponentially decreasing L^s-risk.

In Section 3 we estimate Besov norms in terms of isoperimetric ratios of images, i.e., by the relative number of pixels on the boundaries of a binary image in order to compare the averaging procedures of Section 2 with the more refined estimators defined later.

In Section 4 we introduce an improved version of the threshold estimator defined by Donoho and Johnstone (1992, 1994a,b, 1995).

The so called *hierarchical threshold* estimator checks the coefficients of Haar basis decomposition of a binary image at every level comparing it to some cutoff value determined by large deviations approximations. These tests may suggest that at *any* multi-resolution level this coefficient should have been zero which means that the pertaining dyadic square (for this coefficient in the Haar basis) should consist of white or black pixels only. This enforces all coefficients of the Haar basis decomposition at the higher levels (describing the image in this square) to be zero. This comparison allows to reduce the probability of misclassification of pixels outside vicinity of the boundary of a binary image. Those square regions of the image, where the tests lead to an undetermined conclusion, (i.e., which are not overwhelmingly colored black or white) are subsequently processed by the standard procedures like non-linear threshold estimators or linear moving average estimators.

Finally, upper and lower estimates of the L^s-risk for these hierarchic wavelet threshold estimators in terms of Besov semi-norms are compared with the risk of the linear estimators of Section 2.

Acknowledgment. The authors are grateful to V. Bentkus and A. Yu. Zaitsev for help in preparing the paper.

2. Linear estimators for images with regular boundaries

Let $\mathbb{R}^2$ be an Euclidean plane and $\mathbb{Z}^2 \subset \mathbb{R}^2$ the standard lattice of points with integer coordinates. A function $f : x \mapsto f_x : \mathbb{R}^2 \to \mathbb{R}$ will be called an image. The image is called binary if the function f takes values 0 and 1 only. There is an obvious one to one correspondence between binary images f and subsets $A \subset \mathbb{R}^2$ via $A = \{x \in \mathbb{R}^2 : f(x) = 1\}$. Speaking about binary images we shall use both notation—f and A, assuming always that $A = \{x \in \mathbb{R}^2 : f(x) = 1\}$.

In the following we shall consider discrete images. For any positive integer N, we consider a resolution $1/N$, a scaled lattice $\mathbf{Z} = N^{-1}\mathbb{Z}^2$ and denote by Q the unit half-open square $[0,1) \subset \mathbb{R}^2$. Write $\mathbf{Q} = Q \cap \mathbf{Z}$. Here and in the following we do not