BOUNDS FOR THE ACCURACY OF POISSONIAN APPROXIMATIONS OF STABLE LAWS

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Abstract. Stable laws G_α admit a well-known series representation of the type

$$G_\alpha = \mathcal{L}\Big(\sum_{j=1}^\infty \Gamma_j^{-1/\alpha} X_j \Big), \qquad 0 < \alpha < 2,$$

where $\Gamma_1, \Gamma_2, \dots$ are the successive times of jumps of a standard Poisson process, and $X_1, X_2, \dots$ denote i.i.d. random variables, independent of $\Gamma_1, \Gamma_2, \dots$ We investigate the rate of approximation of G_α by distributions of partial sums $S_n = \sum_{j=1}^n \Gamma_j^{-1/\alpha} X_j$, and we get (asymptotically) optimal bounds for the variation of $G_\alpha - \mathcal{L}(S_n)$. The results obtained complement and improve results of A. Janicki and P. Kokoszka, and M. Ledoux and V. Paulauskas. Bound for the concentration function of S_n are also proved.

1. Introduction

The characteristic function g_α of any stable distribution function G_α allows the representation

$$g_\alpha(t) = \exp\Big\{ -c|t|^\alpha \big(1 + i\beta\varphi(\alpha, t)\big) \Big\}, \qquad 0 < \alpha < 2, \tag{1}$$

where $|\beta| \le 1$, and φ is a well-known function (see, for example Samorodnitsky and Taqqu (1994)).

Introduce a sequence $\lambda_1, \lambda_2, \dots$ of independent identically distributed (i.i.d.) random variables with exponential distribution, that is $\mathbf{P}\{\lambda_1 > x\} = e^{-x}$, for $x \ge 0$.

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It is well-known that the sequence $\Gamma_n = \lambda_1 + \cdots + \lambda_n$ defines the successive times of the jumps of a standard Poisson process.

Let $X, X_1, X_2, \ldots$ be a sequence of i.i.d. random variables with common distribution function F such that

$$0 < \mathbf{E}\,|X|^\alpha < \infty \qquad \text{and} \qquad \mathbf{E}\,X = 0 \ \text{ if } \ \alpha > 1.$$

Assume that sequences $\lambda_1, \lambda_2, \ldots$ and $X, X_1, X_2, \ldots$ are independent. The representation

$$G_\alpha = \mathcal{L}\Big(\sum\nolimits_{j=1}^{\infty} \Gamma_j^{-1/\alpha} X_j \Big), \qquad \alpha \neq 1, \tag{2}$$

of stable random variables with distribution G_α by an almost surely convergent series goes back to P. Levy and was revived by R. LePage [LePage (1989, 1981), and LePage, Woodroofe and Zinn (1981)]. Recall, that the scale parameter c in (1) is connected to X by the following relation:

$$\lim_{x\to\infty} x^\alpha \big(1 - G_\alpha(x) + G_\alpha(-x)\big) = c\, \frac{1-\alpha}{\Gamma(2-\alpha)\,\cos \pi\alpha/2} \; = \; \mathbf{E}\,|X|^\alpha.$$

We shall investigate the convergence rates in (2), that is, we shall obtain bounds for $G_\alpha - \mathcal{L}(S_n)$, where S_n denotes the partial sum

$$S_n = \sum\nolimits_{j=1}^{n} \Gamma_j^{-1/\alpha} X_j.$$

General results on series representation of stable laws (including the cases $\alpha = 1$ and $\mathbf{E}\,X \neq 0$) and formulae connecting F with the parameters c and β in (1), can be found, for example, in Samorodnitsky and Taqqu (1994), Th. 1.4.5. Poissonian representations of stable laws or even infinitely divisible distributions [Ferguson and Klass (1972) and Rosinski (1990)] are valid in general Banach space setting. They are useful in treating structure problems related to stable laws [see, e.g., monographs Marcus and Pisier (1981), Ledoux and Talagrand (1991), Samorodnitsky and Taqqu (1994), Janicki and Weron (1994)]. Such representations have been used in resampling (bootstrap) problems in statistics [see, for example, LePage (1992), Kinateder (1992), LePage and Podgorski (1993), LePage, Podgorski and Ryznar (1994)], and in financial mathematics and option pricing models [Rachev and Samorodnitsky (1992), and LePage, Podgorski and Ryznar (1994)].

A more practical application of the representation theorem is the simulation of stable random variables and especially stable processes [Janicki and Weron (1994), Janicki and Kokoszka (1992a)].

Denote

$$\Delta_n = \Delta_n(\alpha, F) = \sup_x \Big| \,\mathbf{P}\left\{ S_n \le x \right\} - G_\alpha(x) \Big|.$$