Numerical approximation of homoclinic chaos

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Abstract

Transversal homoclinic orbits of maps are known to generate a Cantor set on which a power of the map conjugates to the Bernoulli shift on two symbols. This conjugacy may be regarded as a coding map, which for example assigns to a homoclinic symbol sequence a point in the Cantor set that lies on a homoclinic orbit of the map with a prescribed number of humps. In this paper we develop a numerical method for evaluating the conjugacy at periodic and homoclinic symbol sequences. The approach combines our previous method for computing the primary homoclinic orbit with the constructive proof of Smale's Theorem given by Palmer. It is shown that the resulting nonlinear systems are well conditioned uniformly with respect to the characteristic length of the symbol sequence and that Newton's method converges uniformly too when started at a proper pseudo orbit. For the homoclinic symbol sequences an error analysis is given. The method works in arbitrary dimensions and it is illustrated by examples.

Key words. Dynamical systems, numerical methods, homoclinic points for maps, multihumped homoclinic orbits, chaos

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1 Introduction

The original idea in Smale's Homoclinic Theorem was geometric in nature (cf. [9] and [8], [6]). A transversal homoclinic point of a diffeomorphism implies a horseshoe which may be regarded as a compact invariant set on which a power of the map is conjugated to the full shift on two symbols. Since then, more analytically oriented proofs have been developed (see [7] and the references therein) which also generalize to non–diffeomorphisms (cf. [4], [10]). These approaches seem to be better suited for developing numerical approximations of the invariant set and the conjugacy. In this paper we aim at such a method and we will combine the elegant proof of Palmer [7] (which employs the Shadowing Lemma) with our previous method [2] for computing a primary homoclinic orbit.

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In what follows we will outline the basic idea of the paper in non-technical terms and refer to the corresponding sections for the details.

We consider a smooth diffeomorphism f on $\mathbb{R}^k$ with a hyperbolic fixed point $\xi \in \mathbb{R}^k$. We assume that there exists a transversal homoclinic orbit $(x_n)_{n\in\mathbb{Z}}$ based at ξ, i.e.

$$\begin{aligned} x_{n+1} &= f(x_n), \qquad n \in \mathbb{Z} \\ \lim_{n\to\pm\infty} x_n &= \xi \end{aligned}$$

and the stable and unstable manifolds of ξ intersect transversally at x_0, and hence at each point x_n, $n \in \mathbb{Z}$.

In [2] we have analyzed the numerical approximation of such orbits by solving a finite boundary value problem

$$y_{n+1} = f(y_n) \quad , \quad n = n_-, \ldots, n_+ - 1 \tag{1}$$

$$b(y_{n_-}, y_{n_+}) = 0 \tag{2}$$

where $n_\pm$ are assumed to be large and (2) represents a set of k boundary conditions. Our convergence analysis as $n_\pm \to \pm\infty$ applies to two important cases: *periodic boundary conditions* given by

$$b(y_{n_-}, y_{n_+}) = y_{n_-} - y_{n_+} = 0$$

and *projection boundary conditions* of the form

$$b(y_{n_-}, y_{n_+}) = (b_-(y_{n_-}), b_+(y_{n_+})) = 0,$$

where the zero sets of b_- and b_+ are linear approximations to the local unstable and stable manifold of ξ respectively.

According to Palmer [7] the Cantor set Λ in Smale's Theorem can be constructed as follows. Consider the compact hyperbolic set $\Lambda_{THO} = \{\xi\} \cup \{x_n : n \in \mathbb{Z}\}$ and take any $\delta, \varepsilon > 0$ such that δ-pseudo orbits in Λ_{THO} can be uniquely ε-shadowed by f–orbits in $\mathbb{R}^k$ and such that the closed balls $B_\varepsilon(x_0)$ and $B_\varepsilon(\xi)$ do not intersect. Then choose $L \in \mathbb{N}$ so large that $x_{-L}, x_{L+1} \in B_{\delta/2}(\xi)$ holds and consider the orbit segments of length $p = 2L+1$ given by

$$C_0 = (\xi, \ldots, \xi),\ C_1 = (x_{-L}, \ldots, x_L).$$

To any binary sequence $a \in \Sigma_2 := \{0,1\}^{\mathbb{Z}}$, i.e.

$$a = (\ldots, a_{-1}, a_0, a_1, \ldots), \quad a_i \in \{0,1\}, \tag{3}$$

associate the δ-pseudo orbit

$$(\ldots, C_{a_{-1}}, C_{a_0}, C_{a_1}, \ldots) \tag{4}$$

and let $(y_n)_{n\in\mathbb{Z}}$ be its unique ε-shadowing orbit. The conjugacy Ψ is then defined by $\Psi(a) = y_0$ and the set Λ by $\Lambda = \Psi(\Sigma_2)$. By uniqueness of the shadowing construction it is easy to see that

$$\Psi \circ \sigma = f^p \circ \Psi, \quad p = 2L+1$$