LARGE DEVIATIONS OF PRODUCTS OF EMPIRICAL MEASURES AND U-EMPIRICAL MEASURES IN STRONG TOPOLOGIES

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ABSTRACT. We prove a large deviation principle (LDP) for m-fold products of empirical measures and for U-empirical measures, where the state space $(S, \mathcal{S})$ of the underlying i. i. d. sequence of random variables is an arbitrary measurable space. For this LDP we choose a suitable subset $\mathcal{M}_\Phi(S^m)$ of all probability measures on $(S^m, \mathcal{S}^{\otimes m})$ and endow it with a topology, which is stronger than the τ-topology and makes the map $\nu \mapsto \int_{S^m} f\,d\nu$ continuous even for certain unbounded f. An improved form of a LDP for U- and V-statistics is obtained as a particular application. Furthermore, we improve the Gibbs conditioning principle for interacting ensembles of particles.

1. Introduction, statement of results, and applications

Let $\{X_i\}_{i\in\mathbb{N}}$ be a sequence of independent, identically distributed random variables taking values in a measurable space $(S, \mathcal{S})$. For simplicity we assume that they are defined on the product space $(\Omega, \mathcal{A}, \mathbb{P}) = (S^{\mathbb{N}}, \mathcal{S}^{\otimes\mathbb{N}}, \mu^{\otimes\mathbb{N}})$, where μ is the distribution of X_1. For every $n \in \mathbb{N}$ the empirical measure is defined by $L_n = \frac{1}{n}\sum_{i=1}^n \delta_{X_i}$. The large deviations of $\{L_n\}_{n\in\mathbb{N}}$ have been studied in several papers. Sanov [21] considered the problem when $S = \mathbb{R}$ and the space of probability measures $\mathcal{M}_1(S)$ is endowed with the weak topology. Donsker and Varadhan [11] and Bahadur and Zabell [3] (see also Azencott [2]) proved – with different approaches – a large deviation principle for $\{L_n\}_{n\in\mathbb{N}}$ when S is a Polish space with Borel σ-algebra $\mathcal{S}$, again in the weak topology of $\mathcal{M}_1(S)$. Groeneboom, Oosterhoff and Ruymgaart [17] obtained a stronger result in which S is a Hausdorff topological space with Borel σ-algebra $\mathcal{S}$ and $\mathcal{M}_1(S)$ is endowed with the τ-topology, that is, the coarsest topology which makes the maps $\mathcal{M}_1(S) \ni \nu \mapsto \int_S f\,d\nu$ continuous for all f in the space $B(S)$ of bounded, real-valued, $\mathcal{S}$-measurable functions on S. De Acosta [1] proved the large deviation principle for $\{L_n\}_{n\in\mathbb{N}}$ in the τ-topology setting when $(S, \mathcal{S})$ in an arbitrary measurable space.

According to the strong law of large numbers,

$$\mathbb{P}\biggl(\lim_{n\to\infty}\int_S f\,dL_n = \int_S f\,d\mu\biggr) = 1, \qquad f \in L_1(S, \mathcal{S}, \mu), \tag{1.1}$$

hence it should be possible to extend de Acosta's results to an even stronger topology. For topologies generated by a collection Φ of $\mathcal{S}$-measurable functions $f\colon S \to \mathbb{R}$

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with all exponential moments, i. e. $\int_S \exp(\alpha|f|)\,d\mu < \infty$ for all $\alpha > 0$, such an extension of the large deviation principle for $\{L_n\}_{n\in\mathbb{N}}$ is included in our results, see Theorem 1.8 with $m = 1$. For a Polish space S with Borel σ-algebra $\mathcal{S}$, a similar extension of such a large deviation principle was obtained by Wu [26, Section 2.4.4 and Section 4.1]. The collection Φ generating his topology consists of all measurable $f\colon S \to \mathbb{R}$ which are dominated in absolute value by some multiple of a *single* measurable reference function $\psi\colon S \to [1,\infty]$ with $\int_S \exp(\alpha\psi)\,d\mu < \infty$ for all $\alpha > 0$.

1.1. **Motivating examples.** Given an integer $m \ge 2$, we can consider the set $\mathcal{M}_1(S^m)$ of probability measures on the product space S^m, equipped with the product σ-algebra $\mathcal{S}^{\otimes m}$. Naturally, the τ-topology on $\mathcal{M}_1(S^m)$ is defined to be the one induced by $B(S^m)$, the set of all bounded $\mathcal{S}^{\otimes m}$-measurable functions on S^m. A special feature of the τ-topologies on $\mathcal{M}_1(S)$ and $\mathcal{M}_1(S^m)$ is the possible discontinuity of the map $\mathcal{M}_1(S) \ni \nu \mapsto \nu^{\otimes m} \in \mathcal{M}_1(S^m)$. An example suggested by Y. Peres (see [7, Exercise 7.3.18]) shows that this is not an artifact of certain exotic measurable spaces. Indeed, his example shows that for the compact unit interval $S = [0,1]$ equipped with the Borel σ-algebra the product-measure operation $\mathcal{M}_1([0,1]) \ni \nu \mapsto \nu\otimes\nu$ is discontinuous at the Lebesgue measure. The following example may serve as a further illustration that products of "empirical measures" can exhibit a strange behaviour with respect to the τ-topology.

Example 1.2. Let the circle $S = \mathbb{R}/\mathbb{Z}$ be equipped with the Borel σ-algebra $\mathcal{S}$ and let μ denote the Lebesgue–Borel measure on $(S,\mathcal{S})$. For every $x \in \mathbb{R}$ define the shift modulo 1 (or rotation) θ_x on S by $\theta_x(y) = x + y \bmod 1$ for all $y \in S$. Using these, define

$$S \ni \omega \mapsto L_n(\omega) = \frac{1}{2^n}\sum_{i=0}^{2^n-1} \delta_{\theta_{i2^{-n}}(\omega)} \in \mathcal{M}_1(S), \quad n \in \mathbb{N}_0.$$

Since S is compact, it is easy to verify that $\{L_n(\omega)\}_{n\in\mathbb{N}_0}$ and $\{L_n(\omega)\otimes L_n(\omega)\}_{n\in\mathbb{N}_0}$ converge weakly to μ and $\mu\otimes\mu$, respectively, for every $\omega \in S$.

Next we want to show (1.1), which is the strong law of large numbers. Define the sub-σ-algebra $\mathcal{S}_n$ of $\mathcal{S}$ by $\mathcal{S}_n = \{\, A \in \mathcal{S} \mid A = \theta_{2^{-n}}(A)\,\}$ for every $n \in \mathbb{N}_0$. Note that $\mathcal{S}_{n+1} \subset \mathcal{S}_n$ and $E_\mu[f\,|\,\mathcal{S}_n] = \int_S f\,dL_n$ for every $f \in L_1(S,\mathcal{S},\mu)$ and $n \in \mathbb{N}_0$. Therefore, $\{\int_S f\,dL_n\}_{n\in\mathbb{N}_0}$ is a reversed martingale relative to $\{\mathcal{S}_n\}_{n\in\mathbb{N}_0}$ and it converges μ-almost surely to $g \equiv E_\mu[f\,|\,\mathcal{S}_\infty]$ with $\mathcal{S}_\infty \equiv \bigcap_{n\in\mathbb{N}_0}\mathcal{S}_n$, see for example [20, Corollary V-3-12]. Consider the Fourier coefficients $\hat{g}(n) \equiv \int_S g e_n\,d\mu$ for $n \in \mathbb{Z}$, where $e_n(t) \equiv \exp(-2\pi i n t)$. Given $n \in \mathbb{Z}\setminus\{0\}$, there exist $k \in \mathbb{N}_0$ and an odd $l \in \mathbb{Z}$ such that $n = 2^k l$. Since $g \circ \theta_{2^{-(k+1)}} = g$ and $e_{2^k l}\circ\theta_{2^{-(k+1)}} = -e_{2^k l}$, all coefficients $\hat{g}(n)$ with $n \in \mathbb{Z}\setminus\{0\}$ vanish and, therefore, $g = \hat{g}(0) = \int_S f\,d\mu$ μ-almost surely [12, Chap. 1.5, Theorem 1].

To show that the product measures $\{L_n\otimes L_n\}_{n\in\mathbb{N}_0}$ can go astray, consider the $\mathcal{S}\otimes\mathcal{S}$-measurable set $A \equiv \{\,(x,y) \in S^2 \mid x - y \in \mathbb{Q}\,\}$. By Fubini's theorem, $(\mu\otimes\mu)(A) = 0$. On the other hand, the support of $L_n(\omega)\otimes L_n(\omega)$, which is $\{\,(\theta_{i2^{-n}}(\omega), \theta_{j2^{-n}}(\omega)) \mid i,j \in \{0,1,\dots,2^n-1\}\,\}$, is contained in A for every $n \in \mathbb{N}_0$ and $\omega \in S$. Therefore, the strong law of large numbers does not hold for the $\mathcal{S}\otimes\mathcal{S}$-measurable indicator function 1_A.

The preceding example makes use of the heavy dependence of $\theta_{i2^{-n}}(\omega)$ for $i \in \{0,\dots,2^n-1\}$. We want to show that in the opposite case, where $L_n = \frac{1}{n}\sum_{i=1}^n \delta_{X_i}$