Dualizing rings and a characterization of finite representation type [1]

Henning Krause

Abstract: Using the relation between indecomposable endofinite right and left Λ-modules, we introduce (right) dualizing rings and show that a ring Λ is of finite representation type if and only if Λ is right dualizing and every right Λ-module is a direct sum of (finitely presented) indecomposable Λ-modules. We present also criteria for the existence of non-finitely presented indecomposable endofinite modules over a dualizing ring.

Let Λ be a ring of finite representation type, i.e. Λ is right artinian and there is only a finite number of finitely presented indecomposable right Λ-modules. Then it has been shown by Auslander [1] and by Ringel and Tachikawa [7] that every right Λ-module is a direct sum of finitely presented indecomposable Λ-modules. In this note we shall prove the converse for rings Λ which satisfy the following condition:

(D) If M is a finitely presented indecomposable endofinite right Λ-module, then its dual DM is a finitely presented indecomposable endofinite left Λ-module.

Recall that a right Λ-module M is *endofinite* if it is of finite length when regarded in the natural way as a $\Gamma = \mathrm{End}_\Lambda(M)^{\mathrm{op}}$-module. If M is finitely presented indecomposable and endofinite, we denote by DM the left Λ-module $\mathrm{Hom}_\Gamma(M, I)$, where I is an injective envelope of the unique simple Γ-module. Note that DM is automatically indecomposable and endofinite. We call Λ *right dualizing* if the condition (D) is satisfied. If Λ and Λ^{op} both satisfy (D), we call Λ *dualizing*. We include some examples.

(1) Any ring Λ of finite representation type is dualizing since Λ^{op} is also of finite representation type.
(2) If Λ is a noetherian algebra, i.e. the center $Z(\Lambda)$ is noetherian and Λ is a finitely generated $Z(\Lambda)$-module, then Λ is dualizing [3, Proposition 7.1].
(3) Let Λ be a two-sided artinian ring. If Λ is a PI-ring or, equivalently, if $\Lambda / \operatorname{rad} \Lambda$ is an artin algebra, then Λ is dualizing [8].

These examples show that the following theorem generalizes results of Auslander [2] and Herzog [5].

Theorem *For a ring Λ the following conditions are equivalent:*

[1] To appear in: C. R. Acad. Sci. Paris

(1) Λ *is of finite representation type.*

(2) Λ *is right dualizing and every right* Λ*-module is a direct sum of (finitely presented) indecomposable* Λ*-modules.*

The proof of this result is based on an analysis of the category $(\mathrm{mod}(\Lambda^{\mathrm{op}}), \mathrm{Ab})$ of all additive functors from the category of finitely presented right Λ^{op}-modules to the category Ab of abelian groups, and we shall assume that the reader is familiar with the localization theory for such a Grothendieck category which may be found in [4]. The author would like to thank M. Prest for pointing out that an alternative proof can be obtained by using ideas from [5].

Throughout we fix a ring Λ and we denote by $\mathrm{Mod}(\Lambda)$ the category of (right) Λ-modules. The fully faithful functor

$$\mathrm{Mod}(\Lambda) \to D(\Lambda) = (\mathrm{mod}(\Lambda^{\mathrm{op}}), \mathrm{Ab}), \quad M \mapsto M \otimes_{\Lambda} -$$

is fundamental for our approach. It identifies the pure-injective Λ-modules with the injective objects in $D(\Lambda)$. The category $D(\Lambda)$ is locally coherent, i.e. the subcategory $C(\Lambda) = \mathrm{fp}(\mathrm{mod}(\Lambda^{\mathrm{op}}), \mathrm{Ab})$ of finitely presented functors is abelian, and we can reformulate the conditions of the theorem as follows.

Lemma 1 (1) Λ *is of finite representation type iff* $D(\Lambda)$ *is locally finite, i.e. every object in* $C(\Lambda)$ *is of finite length.*

(2) *Every* Λ*-module is a direct sum of (finitely presented) indecomposable* Λ*-modules iff* $D(\Lambda)$ *is locally noetherian, i.e. every object in* $C(\Lambda)$ *is noetherian.*

Proof: (1) is shown in [1] and (2) follows from the well-known fact that a functor category is locally noetherian iff every injective object decomposes into a direct sum of indecomposable objects.

Remark *A ring* Λ *is said to be right pure-semisimple provided that every right* Λ*-module is pure-injective. An equivalent property is that* $D(\Lambda)$ *is locally noetherian.*

To use the condition (D) we recall the following result which is due to Crawley-Boevey.

Lemma 2 *Let* M *be a finitely presented indecomposable endofinite* Λ*-module. If* DM *is finitely presented, then* $M \otimes_{\Lambda} -$ *is the injective envelope of a simple functor in* $D(\Lambda)$.

Proof: The assumption implies that $M \cong DDM$ by [3, Proposition 6.3]. Thus $M \otimes_{\Lambda} -$ is the injective envelope of a simple functor by [3, Theorem 2.3].