An axiomatic description of a duality for modules

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Given a ring Λ (associative with identity) there is no general procedure known which relates right and left Λ-modules to each other. However it is reasonable to ask for such a relation at least for certain classes of modules. Let us mention three properties which one might expect if one assigns to a right (left) Λ-module M a left (right) Λ-module DM.

(D1) $DDM \cong M$.

(D2) If M is indecomposable, then DM is indecomposable.

(D3) If I denotes an injective cogenerator for the centre C of Λ, then DM is isomorphic to a direct summand of $\operatorname{Hom}_C(M, I)$.

Take for instance a k-algebra Λ over a field k. Then the usual k-dual $DM = \operatorname{Hom}_k(M, k)$ of a finite dimensional Λ-module M satisfies all three properties. However for infinite dimensional Λ-modules the condition (D1) does not hold, and for most rings there is even on the level of the finite length modules no duality functor between right and left modules available.

The aim of this note is to describe without any restriction on the ring Λ a bijection between certain classes of right and left Λ-modules which satisfies (D1) – (D3) and which is uniquely determined by these properties. This bijection covers the k-duality mentioned above but usually there will be non-finitely generated modules on which D is defined. Our approach gives a new interpretation of a construction which is known as elementary duality amongst model theorists, and which is due to Herzog [6], see also [9].

We formulate now the main result. To this end denote by $\operatorname{Mod}(\Lambda)$ the category of (right) Λ-modules and by $\operatorname{mod}(\Lambda)$ the full subcategory of all finitely presented Λ-modules. We shall identify the category of left Λ-modules with $\operatorname{Mod}(\Lambda^{\mathrm{op}})$. Recall that a Λ-module M is pure-injective if every pure monomorphism starting in M splits. A map $\varphi\colon M \to N$ is a pure monomorphism if $\varphi \otimes_\Lambda X\colon M \otimes_\Lambda X \to N \otimes_\Lambda X$ is a monomorphism for every X in $\operatorname{mod}(\Lambda^{\mathrm{op}})$. We call an indecomposable pure-injective Λ-module M *simply reflexive* provided that there is a map $X \to Y$ in $\operatorname{mod}(\Lambda)$ such that the cokernel of the induced map $\operatorname{Hom}(Y, M) \to \operatorname{Hom}(X, M)$ is simple when it is regarded in the natural way as an $\operatorname{End}_\Lambda(M)^{\mathrm{op}}$-module. We refer to the remark below for examples of simply reflexive modules.

Suppose now that M is an indecomposable pure-injective Λ-module which is simply reflexive.

Proposition *There exists, up to isomorphism, a unique indecomposable pure-injective Λ^{op}-module DM such that any map φ in $\operatorname{mod}(\Lambda)$ induces an epimorphism $\operatorname{Hom}_\Lambda(\varphi, M)$ if and* *only if it induces a monomorphism $\varphi \otimes_\Lambda DM$. The module DM is again simply reflexive and satisfies $DDM = M$.*

This has been shown in [8]. However, we include a short proof at the end of this note. We point out that M is simply reflexive if and only if it is reflexive in the sense of Herzog [6] and that DM coincides with the dual of M in the sense of Herzog [6]. The following result describes an essential property of the module DM.

Theorem *Let Γ be a ring and suppose that M is a Λ-Γ-bimodule. Suppose also that I is an injective cogenerator for* $\mathrm{Mod}(\Gamma)$. *Then DM is isomorphic to a direct summand of the Λ^{op}-module* $\mathrm{Hom}_\Gamma(M, I)$.

To formulate an axiomatic description of the module DM we denote by $\mathrm{Ref}(\Lambda)$ the isomorphism classes of indecomposable pure-injective Λ-modules which are simply reflexive.

Corollary *The assignment $M \mapsto DM$ defines a bijection between* $\mathrm{Ref}(\Lambda)$ *and* $\mathrm{Ref}(\Lambda^{\mathrm{op}})$ *which satisfies the conditions* (D1) – (D3). *Moreover any assignment between right and left Λ-modules satisfying* (D1) – (D3) *sends $M \in \mathrm{Ref}(\Lambda)$ to DM.*

Remark 1 An assignment between right and left Λ-modules satisfying (D1) – (D3) is only possible for pure-injective modules. This follows from Lemma 4 below. Note that Λ is a ring of finite representation type iff every right or left Λ-module is pure-injective. In this case every Λ-module is a coproduct of indecomposable modules and every indecomposable module is simply reflexive.

Remark 2 Let M be an indecomposable Λ-module with $\Gamma = \mathrm{End}_\Lambda(M)^{\mathrm{op}}$.

(1) If M is Σ-*pure-injective*, i.e. $\coprod_I M$ is pure-injective for any set I, then M is simply reflexive [6, 8]. In particular if Λ is a k-algebra over a commutative noetherian ring k, then M is simply reflexive if M is of finite length over k. However there are usually simply reflexive Λ-modules which are not finitely generated. Take for instance a Dedekind domain Λ and let $\Lambda_{\mathfrak{p}^\infty}$ be the Prüfer module corresponding to a prime ideal $\mathfrak{p}$. Then $D\Lambda_{\mathfrak{p}^\infty}$ is the corresponding $\mathfrak{p}$-adic module. Note that $D\Lambda_{\mathfrak{p}^\infty}$ is not Σ-pure-injective.

(2) Suppose that M is *endofinite*, i.e. of finite length when regarded in the natural way as a Γ-module. Then M is Σ-pure-injective [3] and therefore simply reflexive. In fact if M is pure-injective and simply reflexive, then M is endofinite iff M and DM are Σ-pure-injective.

(3) Let I be a minimal injective cogenerator for Γ. Then the Λ^{op}-module $\mathrm{Hom}_\Gamma(M, I)$ might be called the *local dual* of M. If M is finitely presented, then it is well-known that