A Construction of Endofinite Modules

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Let k be a field, let Λ be a finite dimensional k-algebra. Krause [K] has considered Λ-modules which are obtained as direct limits of an increasing chain of finite dimensional Λ-modules and which are similar to the Prüfer groups occurring in abelian group theory. He has shown that the Ziegler closure of such a module X always contains an indecomposable endofinite module of infinite length, a socalled "generic" module. This is of importance since the "generic" modules seem to parametrize the families of finite length modules, and families of finite length modules are one of the main objects of present concern [C]. An essential condition used by Krause is the existence of a locally nilpotent endomorphism of X with kernel of finite length. In his proof, Krause uses functor categories as well as model theoretical considerations. The aim of the present note is to present a direct approach (using only modules and their elements) in order to recover and strengthen his result. The author is endebted to Krause for many stimulating remarks.

Let φ be an endomorphism of the Λ-module X. Then φ induces an injection of $\operatorname{Ker}\varphi^{t+1}/\operatorname{Ker}\varphi^{t}$ into $\operatorname{Ker}\varphi^{t}/\operatorname{Ker}\varphi^{t-1}$, for $t \geq 1$. In case $\operatorname{Ker}\varphi$ is finite dimensional, it follows that the vector spaces $\operatorname{Ker}\varphi^{t}/\operatorname{Ker}\varphi^{t-1}$ have the same dimension, for almost all t.

Given a module M and any set I, we denote by M^I the product of copies of M which are indexed by the elements of I, and $M^{(I)}$ denotes the corresponding direct sum of these copies (the module M^I may be constructed as the set of functions $I \to M$, and $M^{(I)}$ is the submodule of all functions with finite support).

Theorem 1. *Let X be an infinite dimensional Λ-module, let φ be a locally nilpotent endomorphism of X with kernel of finite length. Let I be some infinite set. Then the product X^I is the direct sum of a non-zero module $\overline{P}$ of finite endolength and of copies of X.*

The endolength of $\overline{P}$ is bounded by the minimum of the dimension of the vector spaces $\operatorname{Ker}\varphi^{t}/\operatorname{Ker}\varphi^{t-1}$, *with $t \geq 1$.*

Note that if X is a Λ-module and φ is a locally nilpotent endomorphism of X with kernel of finite length, then X satisfies the descending chain condition when considered as a $k[\varphi]$-module and thus also when considered as a module over $\mathrm{End}_\Lambda(X)$. As a consequence, X is Σ-algebraically compact and therefore X as well as all the powers X^I can be written as direct sums of modules with local endomorphism rings [JL, 8.1, 8.2]. Also, if we fix a direct decomposition of a Σ-algebraically compact module $M = \bigoplus_i M_i$ where the M_i are modules with local endomorphism rings, then Azumaya's Theorem (which generalizes the Theorem of Krull-Remak-Schmidt) shows that any indecomposable direct summand U is isomorphic to one of the modules M_i (note that any direct complement of U is also Σ-algebraically compact and therefore again a direct sum of indecomposables, thus U is one of the direct summands of a decomposition of M into indecomposables).

For any countably generated Λ-module M, let $\mathcal{G}(M)$ be the set of isomorphism classes of those indecomposable direct summands of $M^{\mathbb{N}}$ which are infinite dimensional and of finite endolength (thus "generic" modules in the terminology introduced in [C]).

Note that $\mathcal{G}(M)$ may be empty: take any finite dimensional module N and let $M = N^{(\mathbb{N})}$ be the direct sum of countably many copies of N. Then any product of copies of M is again a direct sum of copies of N, thus $M^{\mathbb{N}}$ has no indecomposable direct summand which is infinite dimensional. Let us stress that the examples of this kind satisfy the usual requirements considered in this paper, provided N is non-zero: the module M is infinite dimensional, and there is the shift endomorphism φ (sending the summand with index i to that with index $i-1$ and mapping the first one to zero); this is a locally nilpotent endomorphism of M and its kernel is N, thus of finite length.

Theorem 2. *Let X be an infinite dimensional Λ-module, and assume that X has a locally nilpotent endomorphism φ with kernel of finite length. Then:*

(a) *Let I be any infinite set. The isomorphism class of a module U belongs to $\mathcal{G}(X)$ if and only if U is indecomposable, isomorphic to a direct summand of X^I, but not isomorphic to a direct summand of X.*

(b) *If X has an indecomposable direct summand which is infinite dimensional, then $\mathcal{G}(X)$ is not empty.*

(c) *The number of isomorphism classes in $\mathcal{G}(X)$ is bounded by the minimum of the dimension of the various vector spaces $\mathrm{Ker}\,\varphi^t/\,\mathrm{Ker}\,\varphi^{t-1}$, with $t \geq 1$.*

(d) *Let U be a finitely generated submodule of X with $\varphi(U) \subseteq U$. Then $\mathcal{G}(X) = \mathcal{G}(X/U)$.*

The last assertion has the following consequence: