Elliptic regularity and essential self-adjointness of Dirichlet operators on $\mathbb{R}^n$

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Abstract

We prove a new regularity result for operators of type $L = \Delta + B\cdot\nabla + c$, on open sets $\Omega \subset \mathbb{R}^n$, provided $B : \Omega \to \mathbb{R}^n$, $c : \Omega \to \mathbb{R}$ satisfy mild integrability conditions. As a consequence we prove that $L = \Delta + \nabla \log \rho \cdot \nabla$ with domain $C_0^\infty(\mathbb{R}^n)$ is essentially self-adjoint on $L^2(\mathbb{R}^n, \rho dx)$, if $\rho \in H^{1,1}_{loc}(\mathbb{R}^n)$ and $\nabla \log \rho \in L^\gamma_{loc}(\mathbb{R}^n, dx)$ for some $\gamma > n$.

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One of the classical problems in mathematical physics is the problem of essential self-adjointness for *Dirichlet operators*

$$L := \Delta + \beta \cdot \nabla,$$

with domain $C_0^\infty(\mathbb{R}^n)$ (:= all infinitely differentiable functions on $\mathbb{R}^n$ with compact support) on $L^2(\mathbb{R}^n, \mu)$, where μ is a measure on $\mathbb{R}^n$ with density $\rho := \varphi^2$, with $\varphi \in H_{loc}^{2,1}(\mathbb{R}^n)$ and $\beta := \nabla\rho/\rho$. (By definition $\beta(x) = 0$ if $\rho(x) = 0$). The results obtained in [1], [8], [9], [10], [13], [19] have been important steps in the investigation of this problem. One motivation to study this problem is that the operator $-L$ is unitary equivalent to the Schrödinger operator $H := -\Delta + V$, $V := \Delta\varphi/\varphi$, considered on $L^2(\mathbb{R}^n, dx)$ (see, e.g., [1], [5]) where dx denotes Lebesgue measure on $\mathbb{R}^n$. The corresponding isomorphism $L^2(\mathbb{R}^n, \mu) \to L^2(\mathbb{R}^n, dx)$ is given by $f \mapsto \varphi \cdot f$. Conversely, if $H = -\Delta + V$ is a Schrödinger operator on $L^2(\mathbb{R}^n, dx)$ with lower bounded spectrum $\sigma(H)$ whose minimum is an eigenvalue E, then the isomorphism above holds for the potential $V - E$ (and $\varphi :=$ the ground state). Since this unitary equivalence only holds for sufficiently regular φ, Dirichlet operators are also sometimes called *generalized Schrödinger* operators. We emphasize that under the above isomorphism in general domains change drastically. Hence known results on the essential self-adjointness of H with domain $C_0^\infty(\mathbb{R}^n)$ on $L^2(\mathbb{R}^n, dx)$ do not apply. On the contrary in many cases the essential self-adjointness of Dirichlet operators implies this property for Schrödinger operators (see e.g. [15, pp. 217, 218]).

There are basically two different types of sufficient conditions known for the essential self-adjointness of Dirichlet operators: global and local. A typical global condition obtained in [13] is: $|\beta| \in L^4(\mathbb{R}^n, \mu)$ (provided $\rho > 0$ a.e.). The best local condition obtained so far has been found in [19] where ρ has been required to be locally Lipschitzian and strictly positive if $n \geq 2$ (and with even weaker conditions if $n = 1$, cf. Remark 2 below). In particular, this means that β is locally bounded. One of our main results in this paper (cf. Theorem 7 below) says that L is essentially self-adjoint provided that ρ is merely locally bounded and *locally uniformly positive* (cf. below) and $|\beta| \in L_{loc}^\gamma(\mathbb{R}^n, \mu)$ for some $\gamma > n$ (which as we shall show below, is equivalent to $|\beta| \in L_{loc}^\gamma(\mathbb{R}^n, dx)$; cf. Corollary 8). The proof of Theorem 7 is based on an elliptic regularity result (which is the first main result of this paper) giving $H_{loc}^{\gamma,1}$-regularity of distributional solutions of the elliptic equation $L^*F = 0$, where $Lf := \Delta f + \langle B, \nabla f\rangle + cf$. This result is formulated as Theorem 1 below. As a consequence one gets $H_{loc}^{\gamma,1}$-regularity of invariant measures